To be published in the Handbook of the Mathematics of the Arts and Sciences, ed. Bharath Sriraman, Springer Nature, 2026

The Menagerie of Plane-filling Fractal Curves: From Visual Analysis to Number Theory

Jeffrey Ventrella

Jeffrey@ventrella.com

Abstract

Plane-filling fractal curves are possible because of constraints that allow them to visit every point in a planar region. These constraints are also the cause of some curious visual properties. This chapter explores these visual properties as motivation to discover concepts in number theory. A novel categorization scheme is proposed based on the Gaussian and Eisenstein integers in the complex plane, with implied extension to the larger class of cyclotomic integers, where aperiodic curves can be found. Why do we not find self-similar fractals that can be recursively subdivided into 6 similar parts? Integers in the complex plane provide some clues. Families of plane-filling curves can be classified as p*rime, composite, perfect power*, etc. The fractal menagerie presented here is a context for visual and mathematical discovery. Identifying concepts and categories in turn informs the design of more efficient search algorithms to find new plane-filling curves.

Keywords

plane-filling curve, number theory, Gaussian integers, Eisenstein integers, prime numbers, fractals, search algorithm

Introduction

The publication of Benoit Mandelbrot's book, The Fractal Geometry of Nature (Mandelbrot, 1982) marked a turning point: a new branch of geometry was born. Aided by rapid advances in computer graphics and the internet, our collective visual vocabulary expanded to include a menagerie of strange and wonderful shapes.

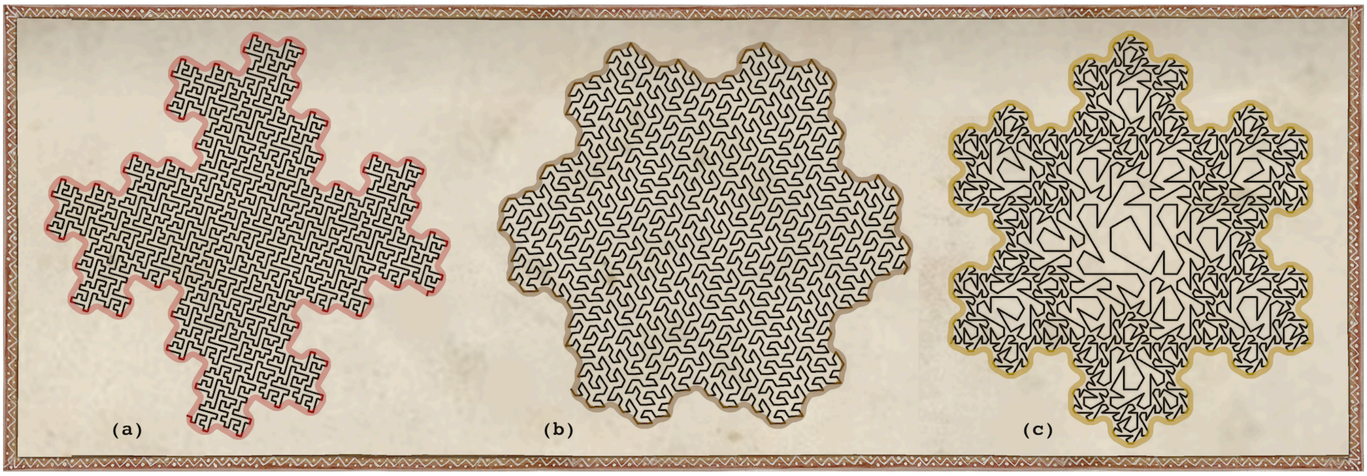

Figure 1. Stylized classics: (a) the Quartet; (b) the Gosper Curve; (c) the Snowflake Sweep

The earliest of these shapes were introduced around the turn of the 20th century. They were considered "mathematical monsters", breaking out of the tidy world of Euclidean geometry. We have much to learn from these monsters. The illustrations in Mandelbrot's book provided an invitation (and an on-ramp of sorts) for the visuallyinclined to venture into this subject. Stunning images of the Mandelbrot and Julia sets took center stage, but many other visual forms were also introduced to a wider audience; notably: *fractal curves*. Figure 1 shows three stylistically-rendered plane-filling fractal curves introduced in the book: (a) the Quartet; (b) the Gosper Curve; and (c) the Snowflake Sweep. Core concepts such as *fractal dimension* were illustrated using diagrams of fractal curve generators. Unlike the curves of Euclidean geometry, fractal curves have infinite detail at all magnifications. They have a fractional (Hausdorff) dimension ranging between 1 and 2. Curves with higher dimension are more "curly" (they fill more *space*). Curves with dimension 2 completely fill regions of the plane. The examples in Mandelbrot's book are just the tip of the iceberg; hundreds of new plane-filling curves have been discovered since. Figure 2 shows several plane-filling curves, some of which are described later. These include (f) (McKenna, 1994); (m) (Fukuda et al, 2001), and (o) (Arndt and Handl, 2019). Three classic curves are shown: (b) the Peano curve, (e) the TerDragon; and (q) the Dragon curve. The other curves shown were discovered by the author (Ventrella, 2012, 2019).

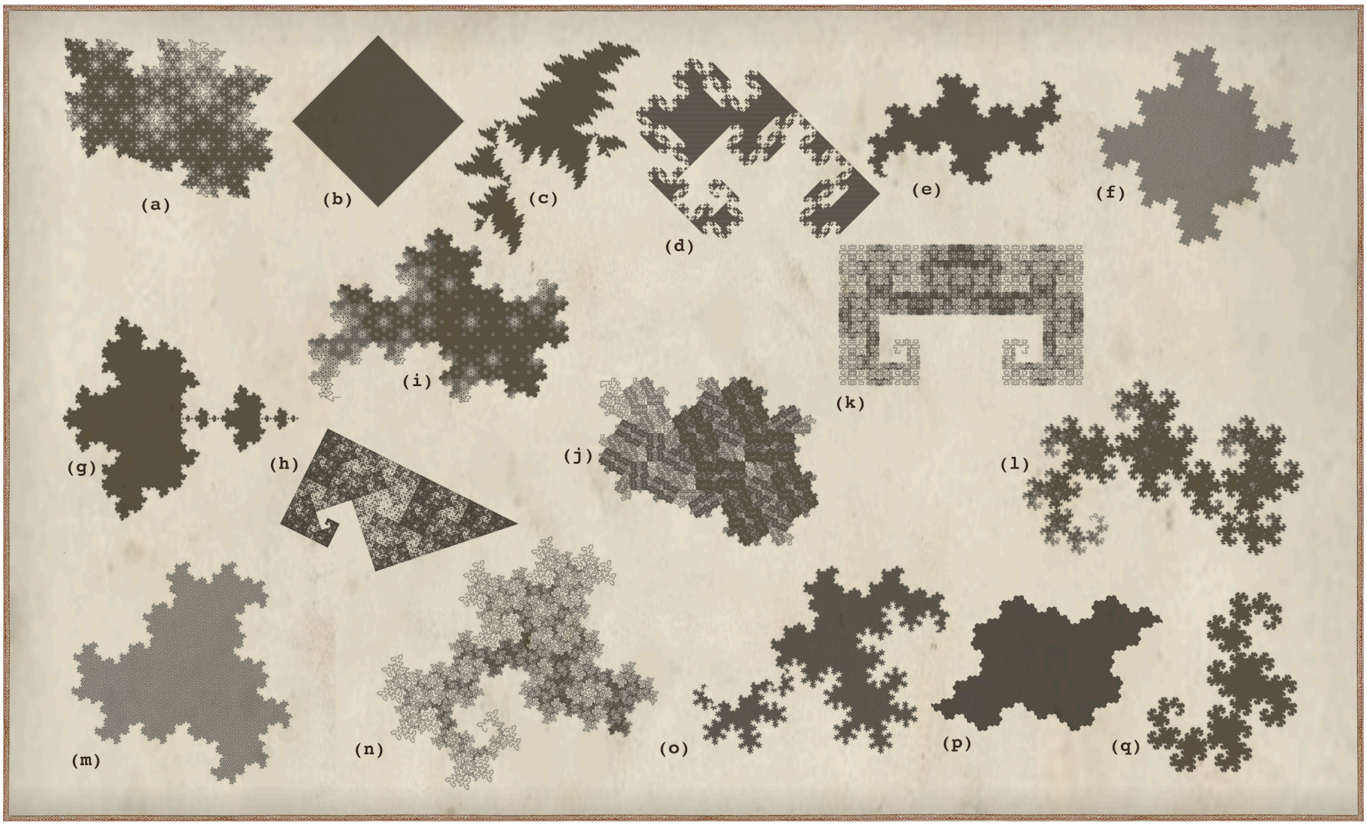

Figure 2. A sampling of plane-filling fractal curves, referenced throughout the chapter.

These curves *sweep* through planar regions, completely filling their interiors. There are endless varieties. Many of the shapes resemble complex forms in nature. They can look strangely familiar…or delightfully alien. What kind of mathematical explanation can we come up with to describe these visual attributes? This chapter proposes a novel classification scheme based on the rings of Gaussian and Eisenstein integers in the complex plane (also referencing the larger class of cyclotomic integers, where aperiodic patterns can be found). These complex integers provide an algebraic framework for exploring their visual properties, and a way to more-optimally search for new plane-filling curves.

Background: Dragons and Other Beasts

A foreshadowing of fractal dragons emerged in mathematics as early as 1917 when Gaston Julia (1918) and Pierre Fatou (1917) were studying iterated functions. They predicted something amazing that would not be seen by human eyes for another 60+ years: Julia sets. An example of a Julia set rendered by Paul Bourke (2001) is shown in Figure 3(d). Julia sets—and the related Mandelbrot set—are generated using iterated functions in the complex plane. Another species of dragons emerges from the sea of Gaussian integers: the square-lattice subset of the complex plane, which has number-theoretic structure, including unique factorization. A few examples are shown in Figures 3(b) (the Twin Dragon) and 3(c) (the Quartet). These fractal shapes are generated using number systems with complex bases (Khmelnik, 1964), (Penney, 1965), (Gilbert, 1982). The classic Dragon curve (Figure 3(a)) was introduced in 1967 (Gardner, 1967). It is often referred to as the "Harter–Heighway" Dragon). Davis and Knuth (1970) showed how the dragon curve can be constructed by interpreting binary numbers in base 1+*i* as drawing instructions. These techniques express the recursive nature of positional numbering systems in the 2D complex plane. Figure 3(e) shows a geometric expression of the Twin Dragon designed by Robert Fathauer (2013) entitled "Twenty Generations of Dragons". Figure 3(f) shows a color rendering of a plane-filling curve discovered by the author (Ventrella, 2019). Its generator is shown later, in Figure 6(b).

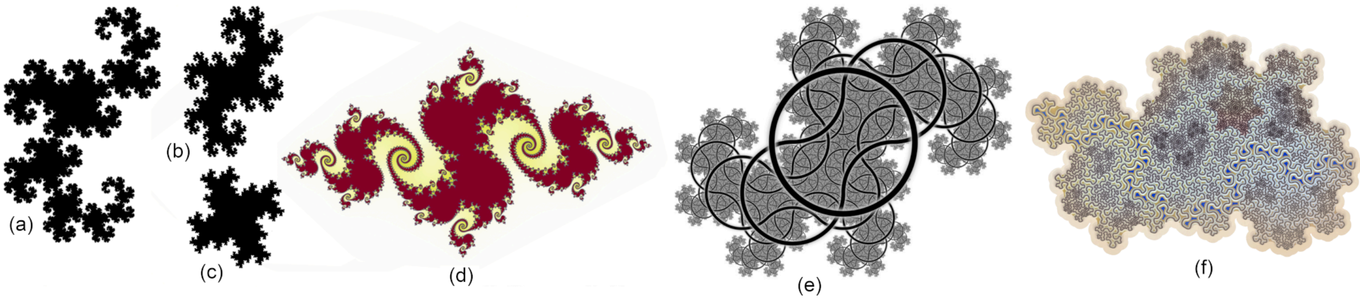

Figure 3. (a) The Dragon curve; (b, c) Fractals in the Gaussian integers generated from complex bases; (d) A Julia set; (e) A design by Robert Fathauer; (f) A stylistically-rendered curve discovered by the author.

The Bestiaries of the Middle Ages depicted many fantastical beasts, often accompanied by moralistic tales with religious themes. The modern dragon-like beasts presented here are not concerned with morals or metaphysics. The tales they tell are about mathematical discovery…but in reference to the mythical beasts of Medieval lore that once filled imaginations, several illustrations in this chapter are rendered in a style reminiscent of illuminated manuscripts.

What does it mean to "discover" a plane-filling fractal curve? Are these curves not already human artifacts? Not entirely. Consider the clever way the curve in Figure 4 grows through recursion, avoiding collision with its own path at every iteration. This behavior was not determined in detail through top-down human design. (Nor was it created by the Medieval dragon in the first panel). Paleontologists discover fossils, and the information gleaned from these discoveries is used to develop evolutionary theory and refine representations of the tree of life. This knowledge can then be used to optimize the search for more fossils. In the set of all possible curves, plane-filling curves are exceedingly rare. Learning how to dig for fossils and to design efficient digging tools (search algorithms) is the key. Several techniques for finding plane-filling curves have been developed by McKenna (1994, 2024), Dekking (2012), Fukuda, et. al (2001), Arndt (2016), and others.

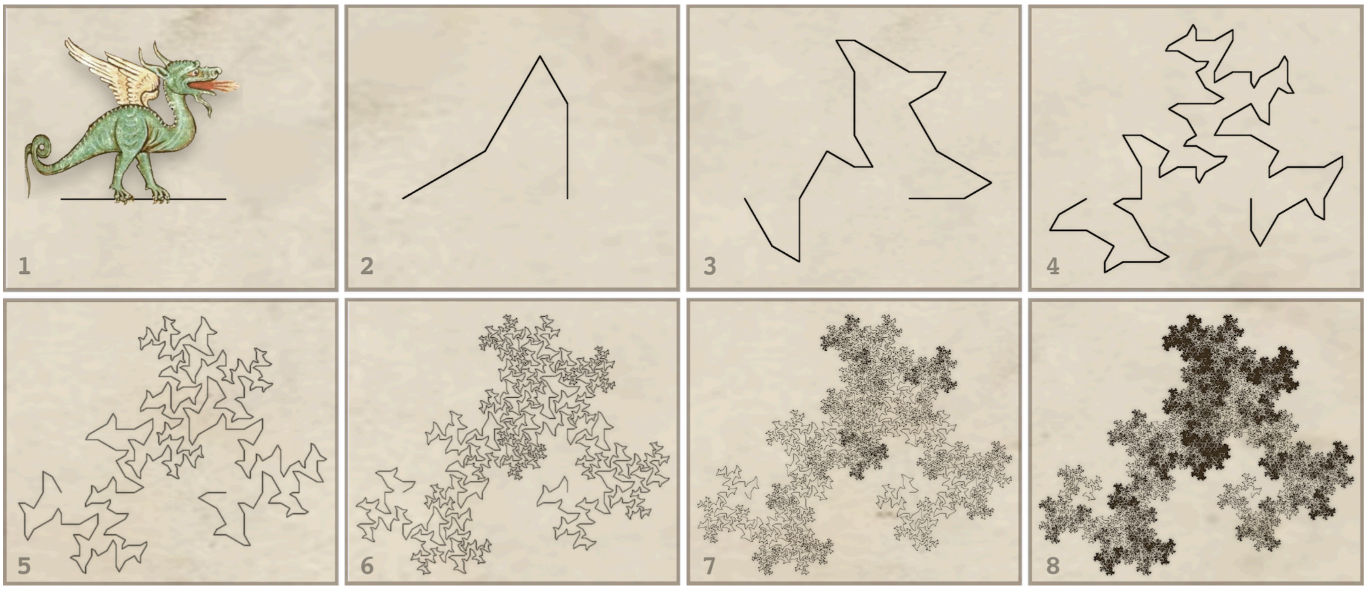

Figure 4. A self-avoiding fractal curve. 1: initiator with Medieval dragon; 2-8: first 7 iterations

Biophilia

For early-human hunter-gatherers, skills in identifying hundreds of plants and animals and generalizing their attributes was critical for survival. We have inherited the curiosity of our ancestors. This is the basis for *biophilia*: an affiliation (usually an attraction but in some cases a repulsion) with natural forms that is the basis for much of our aesthetics (Wilson, 1984). This sensibility translates to the apprehension of mathematical shapes—especially complex shapes generated through fractal recursion. Recursively-defined mathematical objects often exhibit complex organic qualities. This is because they simulate—in an abstract and simplified way—the recursive nature of biological growth, as described by Prusinkiewicz and Lindenmayer (1990). Embryological constraints play out in the development of biological phenotypes. Constraints also play out when fractal curves are generated through recursion. The ability to observe hierarchical structure among families of fractals may be in part due to their generative nature. There is an underlying motivation to hunt for a mathematical basis for these aesthetic qualities.

Discovering a plane-filling fractal curve does not entail a prior and thorough understanding of its mathematical description. There is an element of serendipity due to the emergent complexity resulting from recursion. The *aha* moment of finding a new plane-filling curve often generates motivation to learn and explore more. The discovery process is itself recursive. The reference to biophilia here is to emphasize the apprehension and appreciation of regularities, commonalities, differences, and categories in complex shapes and textures, which are exhibited in plane-filling curves. This approach is compatible with that of Edmund Hariss and Katherine Stange: "…a case-study in the symbiosis of illustration and research, and an entry-point to geometry and number theory for a wider audience" (Hariss, et.al. 2023). Several authors have laid groundwork for the visual appreciation of the mathematics of fractals, including Peitgen and Richter (1986), and Barnsley (1988). Pickover (1990) has elaborated on various ways mathematical formulas can generate organic natural form. It's worth mentioning Dawkins' *Blind Watchmaker* software interface (1986), featuring "biomorphs"—organic shapes made from parameterized fractal trees that can be bred using an interactive evolution algorithm. The experience of breeding biomorphs develops an appreciation for the vastness of the search space, and the value of having rules (whether aesthetic or mathematical) to more efficiently traverse the space.

Plane-filling Curves are not Squares

A "space-filling curve" is typically defined as a mapping from the unit interval [0,1] to the interior of a unit square $[0,1]^2$. Every point on the unit interval maps to a point in the square, and the curve maintains the property of *locality*: nearby points on the interval map to nearby points in the square. The term "space-filling" implies that this concept can be applied to any dimension > 1 (squares, cubes, hypercubes, etc.). This chapter is only concerned with two dimensions, and so the specific term "plane-filling" is used. Giuseppe Peano (1890) is recognized as the first to describe this class of curves. Figure 5(a) shows the original Peano curve, which is based on a 3x3 grid. Figure 5(b) shows some variations of the Peano curve. Bogomolny (2018) has enumerated a large set of Peano Curve variations. The Hilbert Curve (Hilbert, 1891) is based on a 2x2 grid (Figure 5(c). Variations on the Hilbert curve include the Moore curve (Figure 5(d), and the Z-order (or *Lebesgue*) curve (Figure 5(e)). Several families of curves in the square have been identified by McKenna (1994), such as the E Curve (Figure 5(f)).

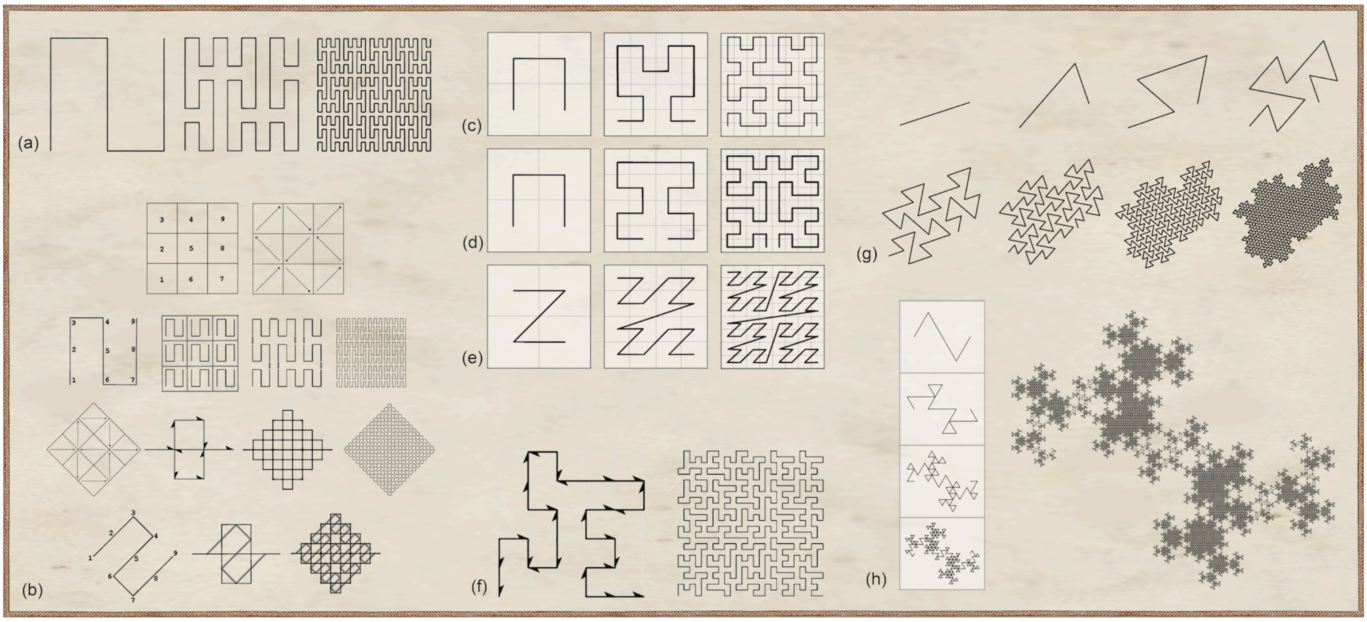

Figure 5(a) original Peano Curve; (b) views/variations of the Peano Curve; (c) Hilbert curve; (d) Moore curve; (e) Z-order curve; (f) E-curve (g) a FASS Curve with a fractal boundary; (h) a curve with a high fractal dimension boundary

The unit square is simple and convenient, but the majority of plane-filling curves do not fill squares, and most of them fill regions of the plane whose boundaries are themselves fractal curves—having their own fractal dimensions and other characteristics. For example, the curve in Figure 5(g) (Ventrella, 2012) is a FASS curve (space **F**illing; self **A**voiding; **S**imple; and **S**elf-similar), having a fractal boundary. Boundaries have been described by Chang, et. al. (2000) who describe the boundary of the Dragon curve. Verrill (2024) developed a system for finding the boundaries of a large class of curves in the square lattice.

The curve in Figure 5(h) has a wild, high-dimensional boundary that appears to be self-contacting at the limit. In the self-contacting case, the notion of "boundary" becomes ambiguous; it is not a *Jordan curve* (a closed curve that divides the plane into inside and outside). This shape takes on a lacunar quality; having holes of multiple scales. This curve is topologically a "pseudo-gasket". (A gasket fractal, e.g., the Sierpinski gasket, has zero area at the limit; it is "all holes", while a pseudo-gasket as defined here has holes but also filled-in areas). McKenna (2019) has explored a class of plane-filling curves whose boundaries have fractal dimensions that can, in theory, be made arbitrarily close to 2.

Another topological type is represented by the Dragon Curve (Figure 2(q)), which is pinched at wasp-waists, making it the topological equivalent of an infinite chain of kissing disks. These monster-like qualities might disqualify these curves from being included in the menagerie if it weren't for the fact that they fit snuggly into this classification scheme: they are self-similar, and at the limit they completely fill one or more *connected* areas of the plane, where the areas are not necessarily the topological equivalent of a disk (or square).

Techniques

The primary technique that will be described here is *edge-replacement*. Analogous to the development of emergent forms in nature, fractal curve growth begins with a *seed* (the fractal generator). The generator is a relatively simple package of information expressed visually as a series of connected line segments. Figure 6(a) illustrates the construction of the Koch curve accomplished by replacing each of the 4 segments ("edges") of the generator with smaller copies of itself. These copies are scaled by 1/3. The process is repeated an *infinite* number of times to generate the *ideal*, Platonic fractal curve (which—paradoxically—has infinite length). For the purposes here, this iterative process only needs to be repeated a handful of times—or at most until the accumulated detail reaches the limit of visual resolution. And for purposes of illustration, the number of iterations (*fractal levels*), is typically set to a small number so that the curve's path can be clearly-seen and fully-appreciated.

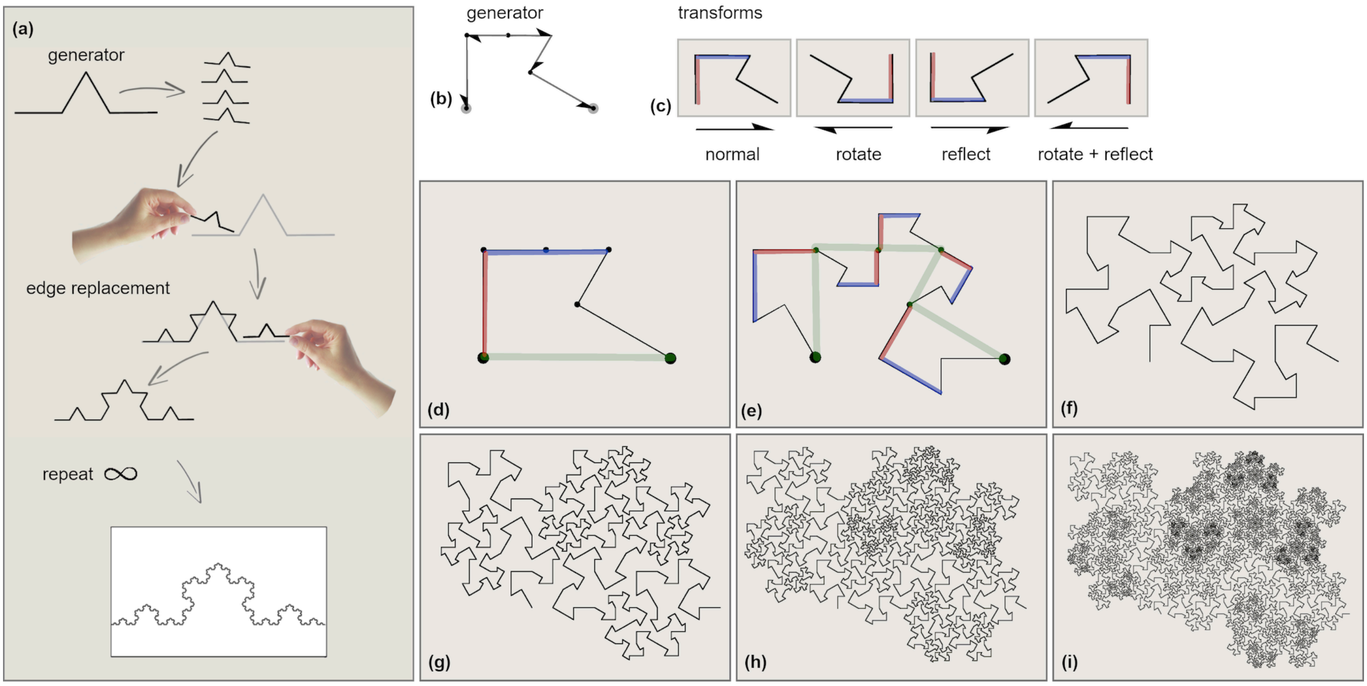

Figure 6. (a) Koch construction; (b) A 5-segment generator; (c) transforms on the generator; (d-i) iterations of the generator.

Edge-replacement is one of several techniques for generating plane-filling curves. It encompasses a large class of curves, and it is intimately related to *node-replacement* (Prusinkiewicz, et. al, 1990) by way of a plane-filling curve's associated tiling. To illustrate this, Figure 7(a) shows how the Peano sweep can be mapped to the Hilbert curve by a translation from edge to node. Figure 7(b) shows how the Gosper curve can likewise map to the "Node Gosper" (Ventrella, 2012) by connecting the centers of the hexagons associated with each edge. Note that node-replacement curves are not strictly self-similar because they require a minimal connecting segment at each iteration, shown in red.

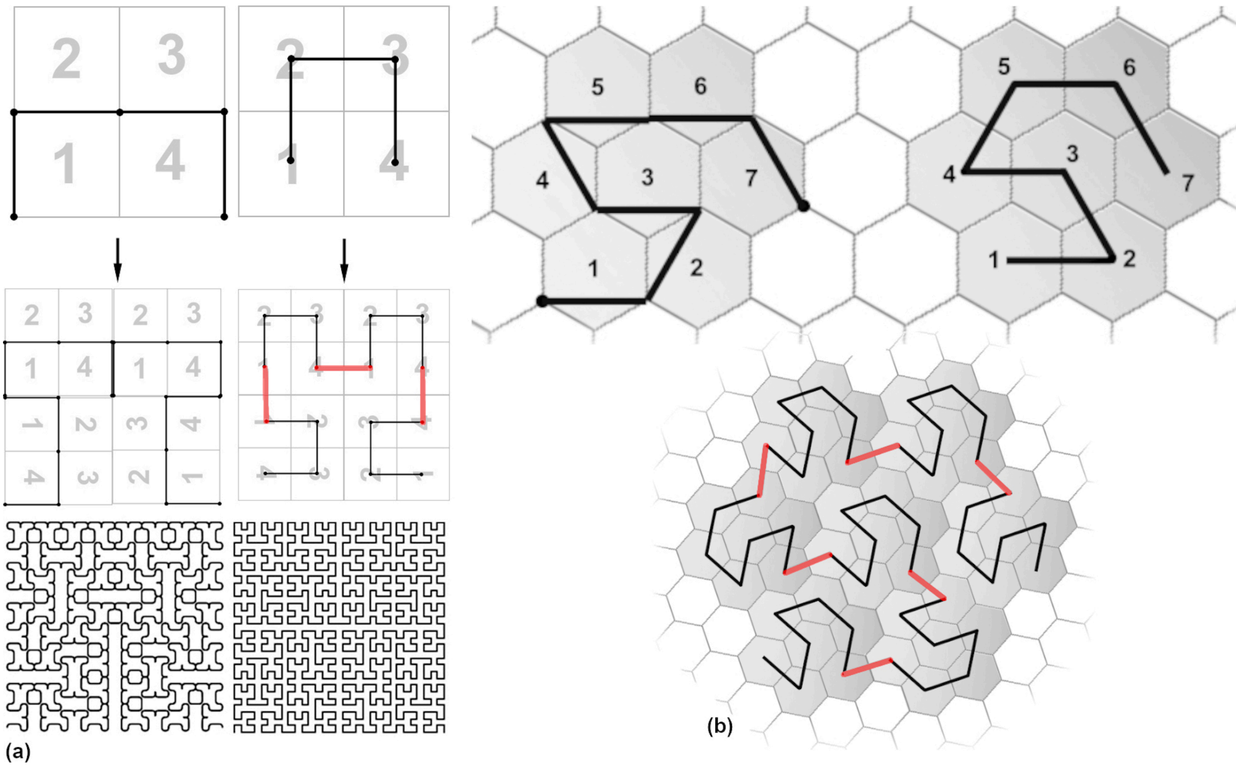

Figure 7: (a) The Peano sweep maps to the Hilbert Curve; (b) The Gosper curve maps to the "Node Gosper".

L-Systems

A technique commonly used to generate fractal curves is the L-System (also called "string rewriting system"), using Turtle Graphics (Abelson et. al., 1986) in a polar coordinate system. The turtle serves as a drawing cursor that can rotate in place and move forwards or backwards, optionally drawing lines. Turtle commands can be represented in L-system strings to generate curves of infinite variety. The L-system starts with a string of symbols that represent turtle drawing instructions for constructing the generator and the rules for recursively replacing certain symbols with variations on the whole string. The string expands in length exponentially at each recursive step to generate the entire set of instructions for drawing the curve at a given resolution. The L-system provides a high level of abstraction: it can be generalized to a variety of geometrical realms (e.g., branching structures, cell division, etc.). The L-system technique is not the focus in this chapter, as the emphasizes is on the properties of a specific integer structure. Edge-replacement can also be described as a special case of an iterated function system (Hutchinson, 1981), in which a geometrical object (in this case a line segment) is replaced with n copies of itself where each copy has a transformation applied (translation, rotation, and scaling).

Puzzling Paths

Fractal curve generators stimulate the puzzle-solving instinct. Consider the 5-segment generator in Figure 6(b). One may be compelled to ask how this generator could be responsible for such a clever self-avoiding curve: it never touches itself or crosses itself, no matter how intricate and detailed it becomes through iteration. One may also wonder why the complex internal pattern in Figure 6(i) emerges (here's a hint: the segments are not all the same length, creating what appears as variations in density within the interior). This is also a property of the Snowflake Sweep (Figure 1(c)) and many other curves shown here.

The rules for generation are relatively straightforward: replace each segment of the curve at each iteration with a scaled-down copy of the generator. In the examples of Figure 6(d-i), the generator must undergo isometric transformations (being rotated by 180 degrees and/or reflected about its main axis, as illustrated with the half-arrows in the generator). Readers are encouraged to work out visually how these progressively detailed curves emerge by replacing each segment with a transformed copy of the generator. Figures 6(d) and 6(e) are decorated with colored lines to help visualize these transforms.

In studying generators that permit plane-filling curves, we see regularities: for instance, some generators have segments connected at angles that are multiples of ±45 degrees; others are multiples of ±30 degrees. These regularities are important for enabling their plane-filling properties. While there are certainly interesting fractal curves with low fractal dimension (e.g., the Koch curve has dimension ~1.2619), the more interesting ones tend to have higher fractal dimensions. A curve with sufficiently-low fractal dimension is like the path of an insect on a long-distance mission: unlikely to come around and cross over its path. The higher the fractal dimension, the more likely (and sooner in its journey) the path will come in contact with itself. One can get a sense of the space of all possible fractal curves by designing generators with random angles, random segment lengths, and a random number of segments. Many of these might be visually interesting, but almost none of them will be plane-filling. It is necessary to have some way to constrain the path. A regular grid (or lattice) provides this constraint.

Complex Integer Lattices

There are two primary kinds of lattices considered, corresponding to the Gaussian integers $\mathbb{Z}[i]$ (illustrated with the square lattice in Figure 8(a)), and the Eisenstein integers $\mathbb{Z}[\omega]$ (illustrated with the triangular lattice in Figure 8(b)).

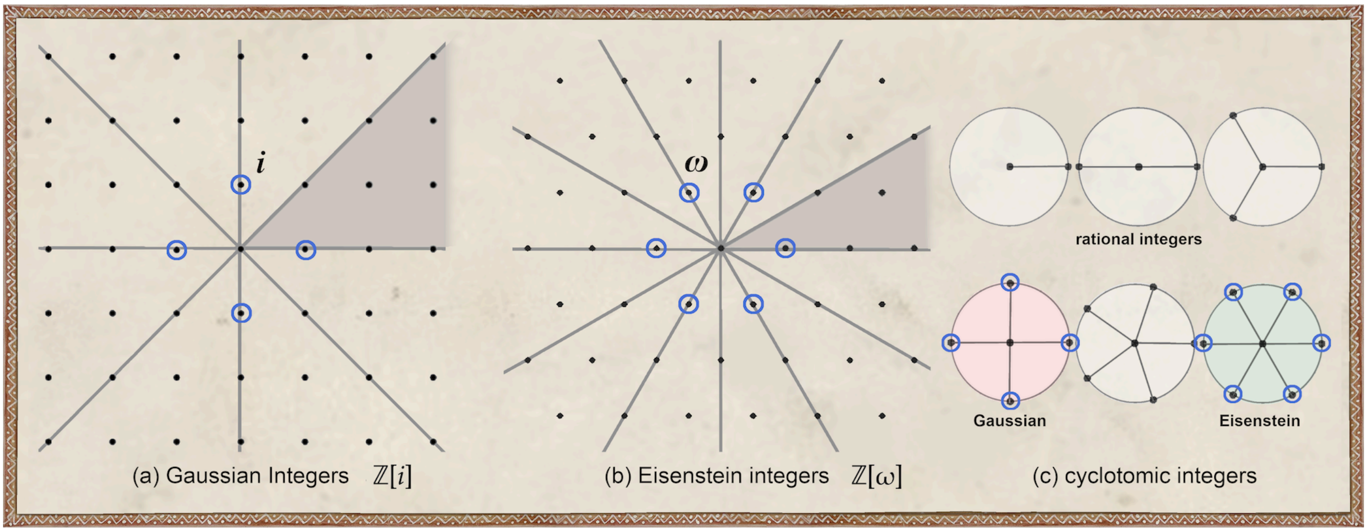

Figure 8. Roots of unity; (a) Gaussian integers; (b) Eisenstein integers; (c) the first 6 cyclotomic fields

A Gaussian integer is a complex number $a+bi$ where the real and imaginary parts are both integers ($a,b \in \mathbb{Z}$). An Eisenstein integer is similar but it requires extra specification to account for its non-orthogonal positioning in the complex plane. It can be specified as $a+b\omega$ where ω is a *primitive cube root of unity*, shown in Figure 8(b). Every Gaussian or Eisenstein integer can be uniquely specified using two rational integers a and b. In the present context, these two rings of integers are sometimes lumped together and referred to simply as *complex integers*. Each forms a *Euclidean domain*: adding or multiplying two Gaussian integers always results in a Gaussian integer, and there are primes, composites, prime powers, etc. The ring of Eisenstein integers has its own set of primes and associated properties. The plane-filling curves that grow in these two kinds of gardens have associated visual qualities.

In Figure 8, the roots of unity are shown with blue circles. There are 4 roots of unity in the Gaussian lattice, and 6 in the Eisenstein lattice. These 2 kinds of complex integers can be seen as members of a more general classification: the *cyclotomic integers*. The roots of unity in the first 6 cyclotomic rings are shown in Figure 8(c). Later it will be shown that the 10th cyclotomic ring can be used to generate a unique class of plane-filling curves with golden ratio proportions and aperiodic patterns, by way of a slight modification to the algorithm. The gray slices shown in Figure 8(a) and (b) extend out indefinitely. These slices are sufficient for referencing all families of plane-filling curves: the other slices have the same relevant properties except they are isometric transformations of the gray slices (reflections and/or rotations). The bottom horizontal edge of the gray slice corresponds to the real number line.

Families

Figure 9 shows these two lattices as grids in perspective, each with 4 example families of plane-filling curves and their associated complex integers (*initiators*) in the lattice. These initiators are labeled with their *norms*. The norm of a complex integer can be described as the square of the Euclidean distance (*modulus*). Norms are always non-negative rational integers. This spatial placement of families is the basis for the categorization scheme. (Note that the Sierpinski arrowhead curve is not a plane-filling curve, having a fractal dimension of ~1.585, but it is a well-known curve and it fits within this classification).

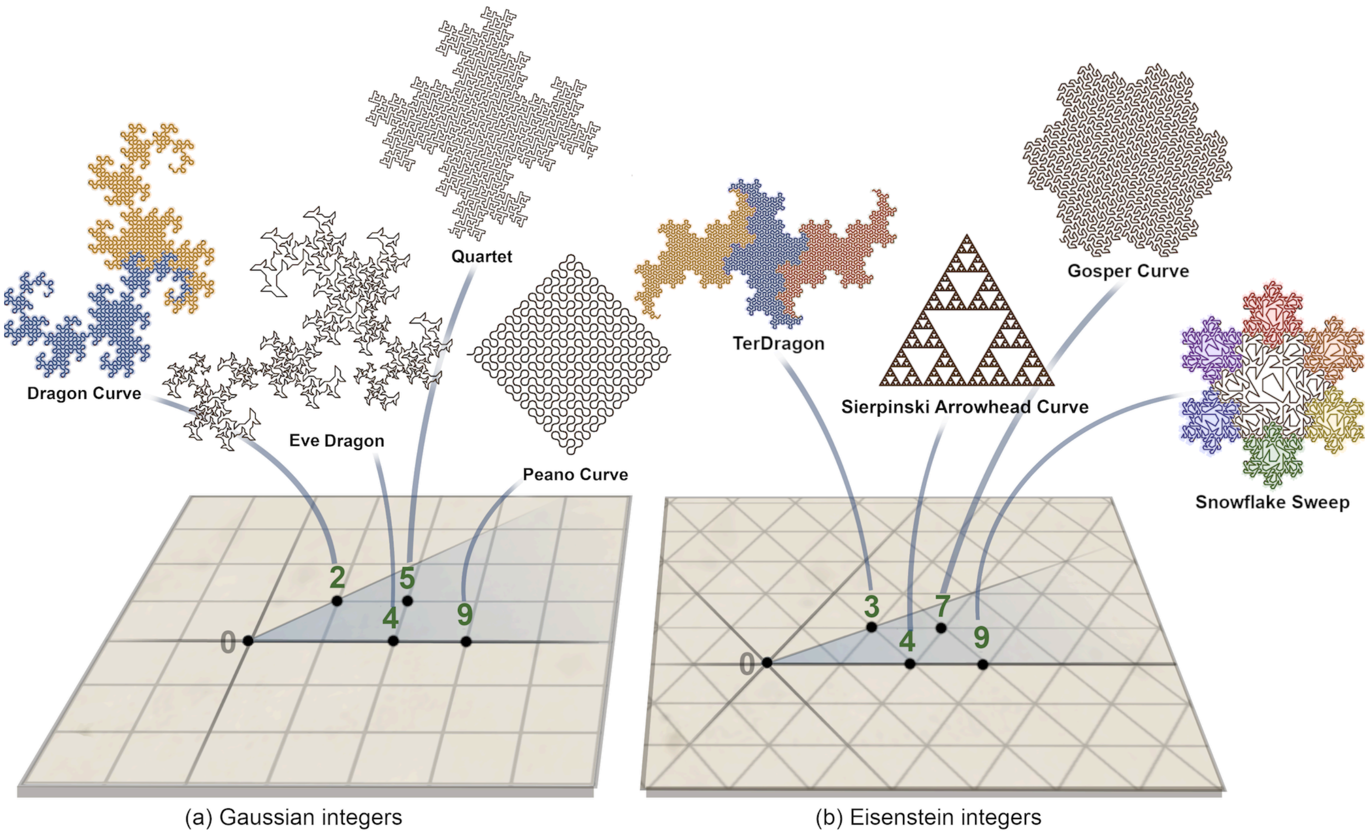

Figure 9. Representative families (labeled by *norm*) of plane-filling curves; (a) Gaussian; (b) Eisenstein.

This way of classifying plane-filling curve families is proposed in (Ventrella, 2019). Consider how these initiators can be broken down into their constituent parts. What are their divisibility characteristics? The techniques described earlier by Davis and Knuth (1970) use complex bases for generating fractals in the Gaussian integers. They reveal the recursive nature of positional numbering systems, which have innate self-similar structure. The present approach achieves self-similar structure using complex addition, multiplication, and conjugation in a recursive function. While the specific algorithm for generating these curves is not the primary focus here, it provides a partial proof of concept: these curves can be generated using complex number operations applied recursively. More importantly: the algorithm is informed by the constraints imposed by the initiators, and this helps reduce the search space for finding new plane-filling curves, as described further below.

The *initiator* of a fractal curve is expressed as a line segment (e.g., the green segments in Figure 6(d, e), and the blue segments in Figure 10). The initiator of a fractal curve is typically described in the literature as the unit interval, and the process of fractal recursion generates progressively smaller copies of the generator. This normalizes all curves to the unit interval. In contrast, the initiator used here is represented as a complex integer and expressed as a vector of length $\sqrt{f}$, where f is the norm of the integer. Assuming these curves are normalized to the space of their initiators, they will fill different areas in the plane, as illustrated in Figure 10, showing 4 examples of plane-filling curves positioned in a square lattice.

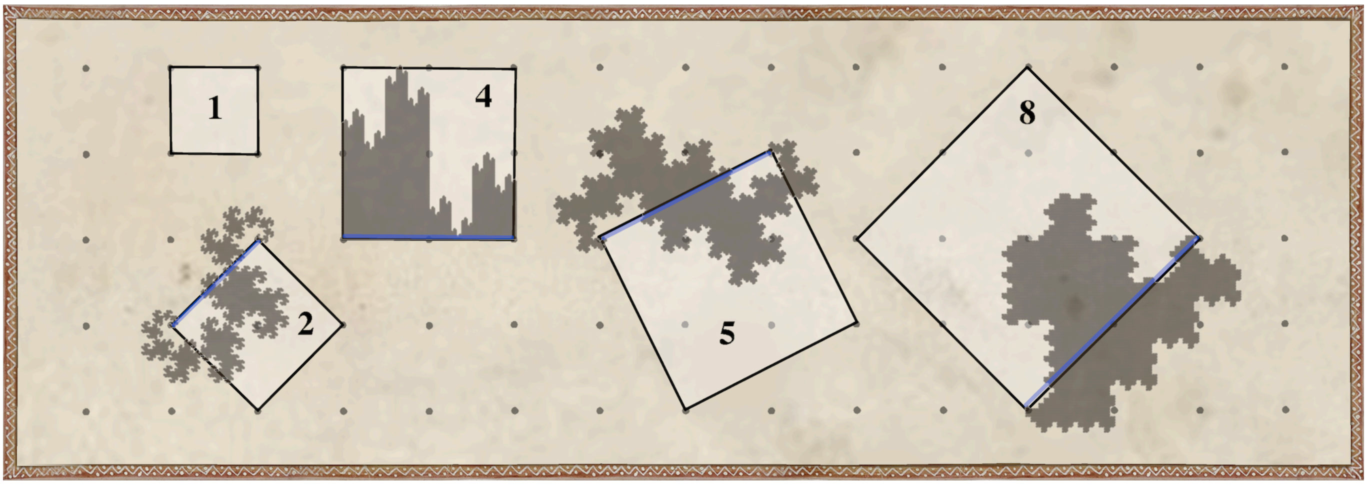

Figure 10. Plane-filling curves filling different areas in the complex plane.

Each plane-filling curve in Figure 10 can be associated with a square whose area is shown. The small area-1 square is included to indicate that that there are no plane-filling curves with norm 1. A conjecture presented here is that the plane-filling curves in Figure 10 fill areas that are *half* of their associated square regions. This is because each of these are *edge-filling* curves; within the boundary they traverse every edge once and visit every lattice point twice. Edge-filling curves are as *tight* or *dense* as a plane-filling curve can be. They are "chunks of lattice" in Mandelbrot's terms. On the other hand, given the same initiator, self-avoiding curves are less dense; they have lower average curvature and fill larger areas.

The Lattice Constrains the Form

There is satisfaction in the simplicity of classifying things in 1-dimensional ordered arrays. In many cases, classifying self-similar fractals by integer *order* (scaling factor) is useful: this refers to the way a self-similar fractal can be subdivided into ***n*** similar parts. For instance, the classic Dragon (Figure 9(a)) can be divided into 2 similar parts, and the TerDragon (Figure 9(b)) can be divided into 3 similar parts, as indicated with coloring. But not every rational integer can be associated with a family of self-similar curves. For instance, there are no plane-filling curves of order 6. (Observe that 6 is not the sum of two squares: a property of the norm of every Gaussian integer). And it turns out that there are also no norms in the Eisenstein integers with a value of 6. The implication is that a self-similar plane-filling curve might be better represented using the *norm* of its associated family initiator. The set of complex integers is not ordinal, and the norms of complex integers do not take every integer value; some integers are missing, and multiple complex integers can share the same norm (e.g., 5 and 3+4*i* both have norm 25). With this in mind, we will sometimes refer to the *order* of a family of plane-filling curves as being equal to its norm, and in some cases the specific integer is required for disambiguation.

Fathauer (2025) made related observations regarding *polyominoes*, *polyiamonds*, and *polyhexes* (polygons made from joining regular polygons edge-to-edge (using either squares, triangles or hexagons). Many of these *polyforms* can be iterated to form *reptiles* (tessellating tiles with fractal boundaries). Those that admit tessellating reptiles appear to have the same scaling factors as the plane-filling curves presented here.

Every plane-filling fractal curve belongs to a unique family, and that family can be identified by its associated initiator ***f***. For instance, in Figure 9(a) the complex integers shown are 1+*i* (norm 2); 2 (norm 4); 2+*i* (norm 5); and 3 (norm 9). Three of these plane-filling curves (excluding the Eve Dragon) belong to prime families (their initiators are Gaussian primes). As prime family curves, they have a special property: every segment in the generator is a unit—analogous to the divisors of rational primes. The Eve Dragon, in contrast, belongs to a *perfect power family*, having norm 4. A Gaussian integer with norm 4 is the product of two integers with norm 2. This is the reason its generator can have variable segment lengths (1 and √2), as shown in Figure 11(a). The Snowflake Sweep (Figure 11(b)) also belongs to a perfect power family, having Eisenstein norm 9: its generator has one segment of length √3, and that segment accounts for the larger central region of the colored Snowflake Sweep in Figure 9(b).

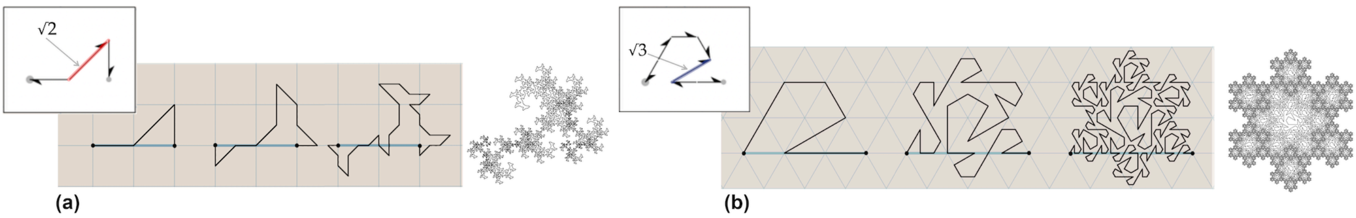

Figure 11. The Eve Dragon and the Snowflake Sweep have multiple-length segments.

The segments of a generator correspond to an ordered array of integers that sum to ***f***. Figure 12 shows the 10 plane-filling curves of the prime order 3 family in the Eisenstein integers, with their generators. These were found through an exhaustive search algorithm. Note that there are two distinct kinds of generator shapes, up to isometries (ignoring reflections and rotations), consisting of 3 units.

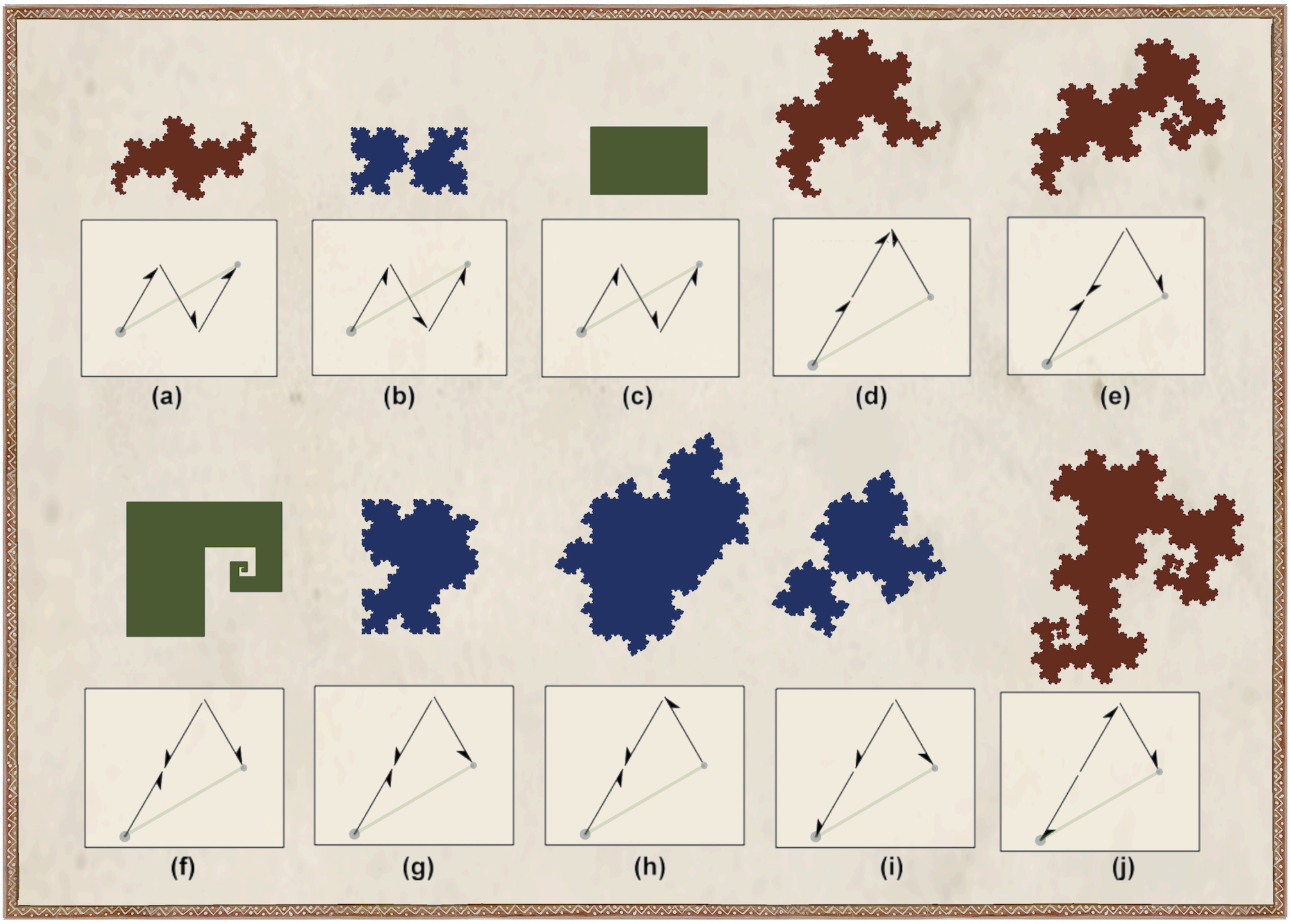

Figure 12. Plane-filling curves and associated generators of the order 3 family.

Notice that Figure 12(h) (also shown in Figure 5(g)) and Figure 12(j) appear larger than the others; these are self-avoiding curves, having the same initiator as the others in its family but being more "spread-out" as shown in Figure 13. In terms of the number of generator segments, these might be considered the *simplest* self-avoiding curves (there are none with 2 segments).

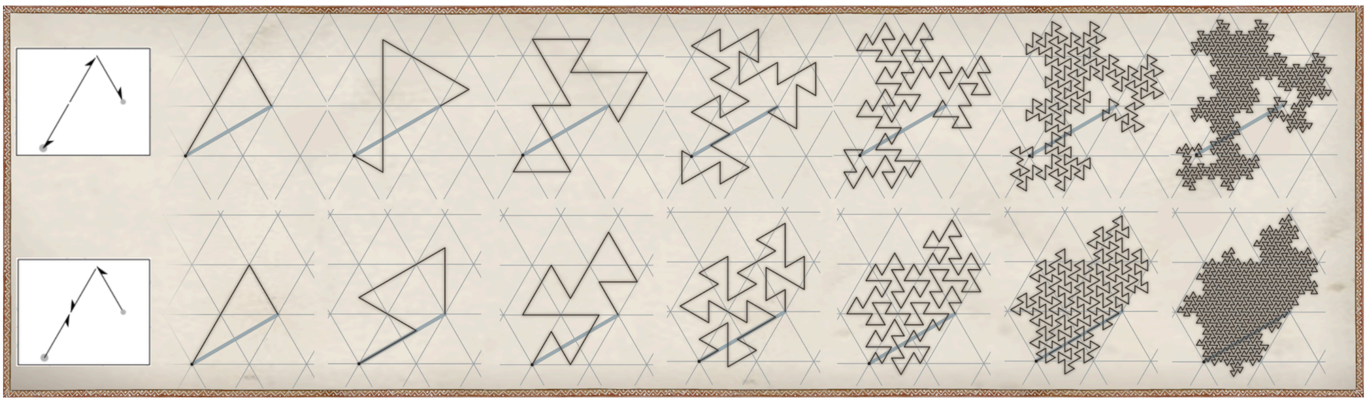

Figure 13. Self-avoiding curves of the Eisenstein order 3 family.

The coloring in Figure 12 indicates that there are three kinds of "skin". Dark green (Figures 12(c) and (f)) indicates *straight* skin (the boundary has fractal dimension 1). Dark red (Figures 12(a), (d), (e), and (j)) indicates "TerDragon skin", having a fractal dimension of 1.2619 (which is curiously the same as the Koch curve). Notice the horn-like protrusions that can be seen in all 4 of these shapes. Dark blue (Figures (b), (g), (h), and (i)) indicates a third kind of skin, with different shaped protrusions. Skin is a curious emergent property: coming into existence through the unique way the curve meanders within its confined space to form the *flesh*. Note that curves having the same skin type are candidates as tessellating *fractiles*.

Figure 14 shows all 12 plane-filling curves of the perfect power order 4 family in the Gaussian integers. The light green color (Figures 14(a), (c), (e), and (g-k)) indicates straight skin. The light purple color (Figures 14(d), (f), and (l)) indicates Dragon skin. The "V-Dragon" (Figure 14(f)) is a curious specimen: its body parts are similar to those of the Dragon curve, except that they are arranged differently. The pink color (Figure 14(b)) indicates Eve-Dragon skin. Several of these generators consist of a combination of units and diagonal norm-2 integers. This accounts for the variation in segment length that accumulates with each iteration (as in Figure 14(g), a curve discovered by Victor Carbajo (2007)). An internal pattern emerges as the curve is progressively iterated. Further iteration makes the pattern more complex. Like the emergent patterns in *skin*, the emergent patterns in *flesh* exert a tug on our visual curiosity: what do these patterns express mathematically?

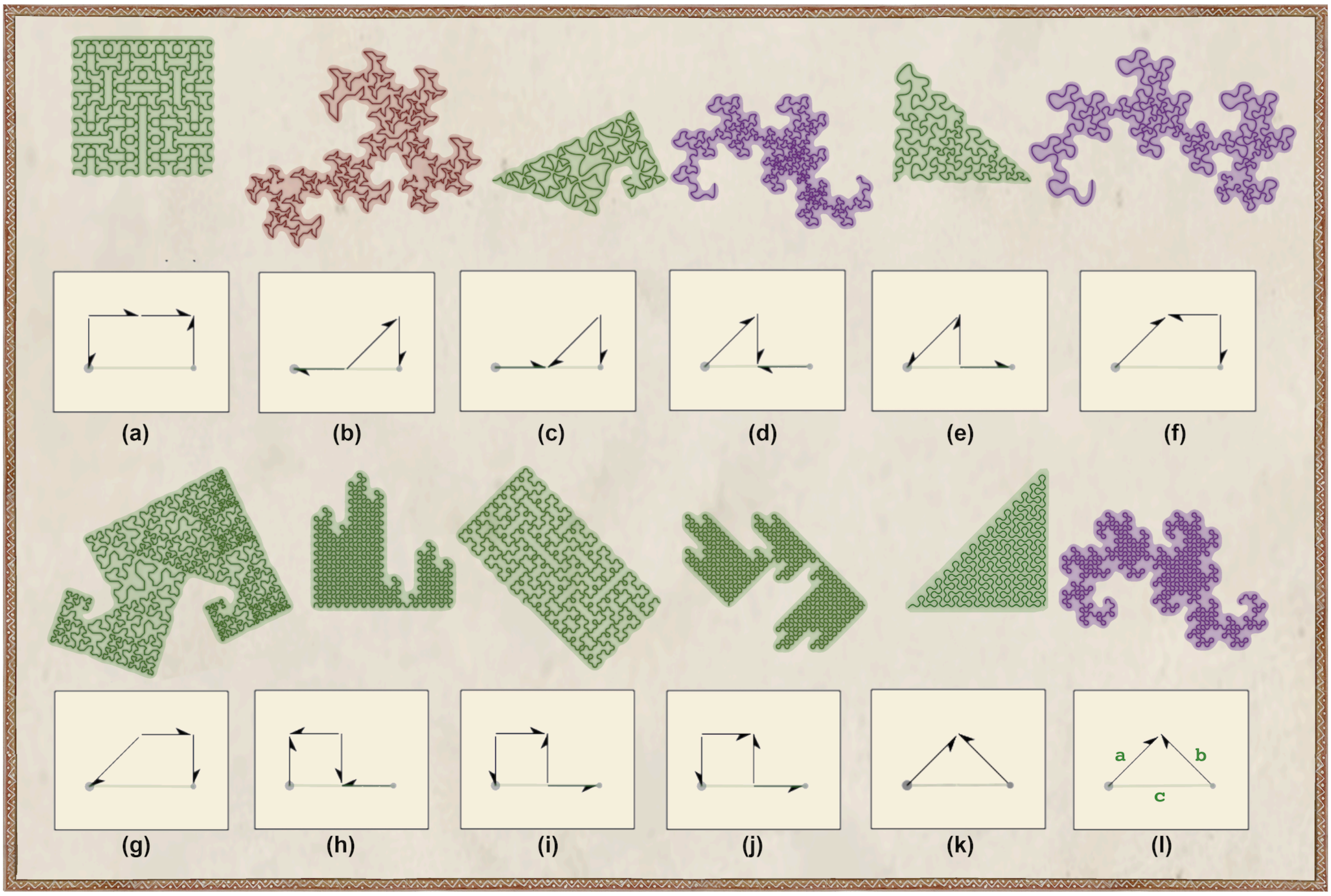

Figure 14. Generators and associated plane-filling curves of the Gaussian order 4 family.

Generalizing the Pythagorean Theorem

The 2-segments of the generator for the "squared" Dragon curve shown in Figure 14(l) forms an isosceles right triangle when combined with its initiator. In a right triangle with hypotenuse c and legs a and b, $a^2 + b^2 = c^2$. You could think of the initiator as the hypotenuse and the generator segments as the legs. This principle applies to all plane-filling curve generators: the norms of the integers (segments) sum to the norm of the initiator.

Here's an exercise: draw a sequence of connected line segments (an ordered array of complex integers) starting from 0 and ending at ***f***, where ***f*** is an arbitrary Gaussian or Eisenstein integer with norm > 1 and < about 10. The following conditions must be met:

1. The path must not cross or overlap with itself. (It may touch itself at points in the lattice).
2. The integers must sum to ***f***.
3. Each integer must either be a unit or an *n*th root of ***f*** (it equals ***f*** when raised to a rational integer power).

An ordered array of integers that satisfies these requirements becomes a *candidate* as a generator for a plane-filling curve. Another requirement is an optional transform to some or all segments. In almost every case for the generators in Figures 12 and 14, transforms (indicated by half-arrows) are required to generate these plane-filling curves. The only curve that does not require any transforms is the TerDragon (Figure 12(a)). Its half-arrows indicate normal non-transformed segments. The majority of random transforms will not suffice. In the case of the Peano sweep generator in Figure 14(a), the number of possible permutations of transforms among its 4 segments is 4^4 = 256. The Gosper curve (Figure 7(b)) has 7 segments, so the number of permutations is 4^7 = 16384.

Perfect Power Family Sets

At the bottom of Figure 15 are the two primitive dragons of the Gaussian and Eisenstein integers: the Dragon Curve and the Ter Dragon. Each is a member of the smallest family. Their initiators are $\boldsymbol{g} = 1+i$, and $\boldsymbol{e} = 2+\omega$, respectively.

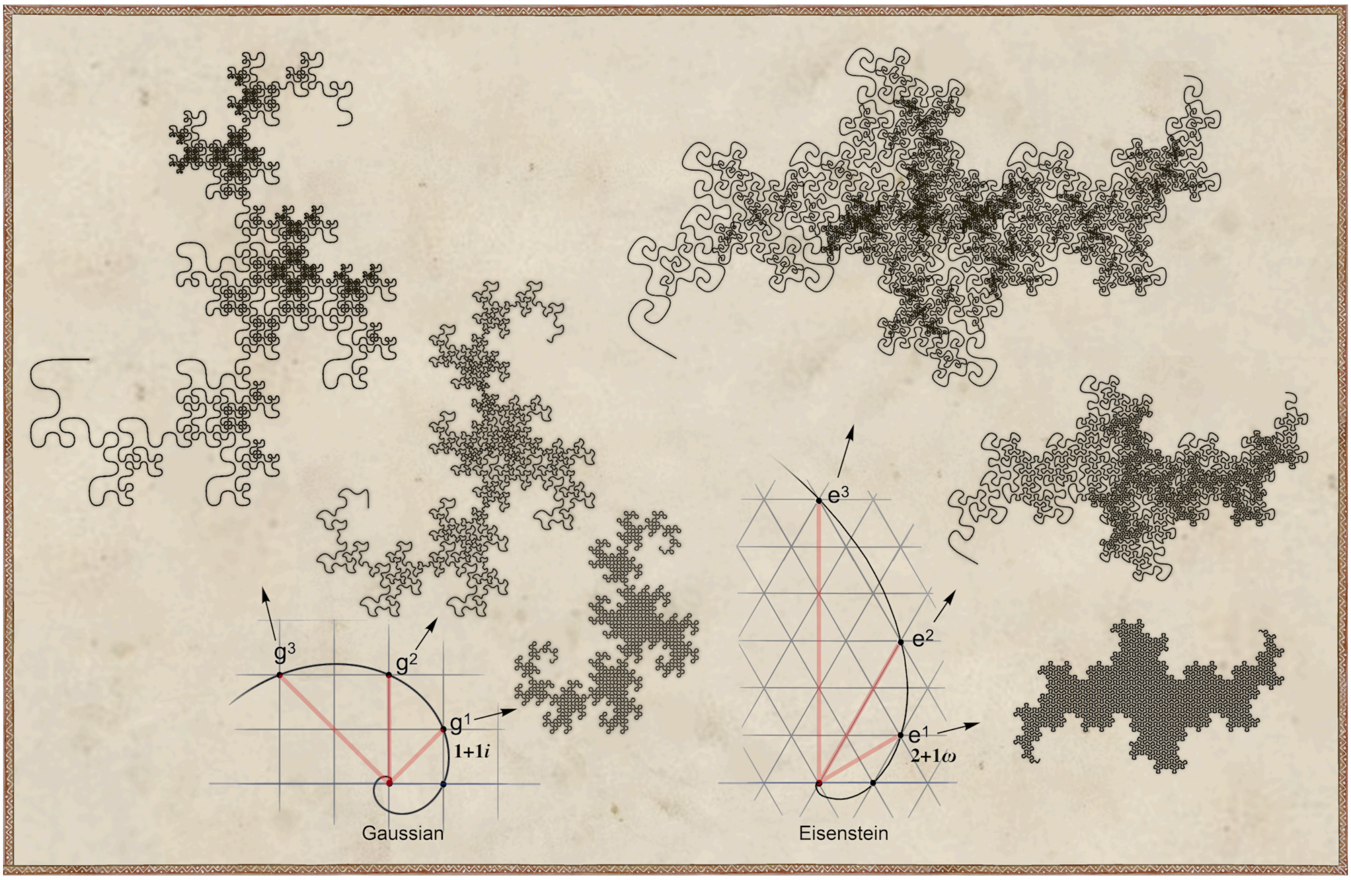

Figure 15. The Dragon Curve and the TerDragon are members of the smallest families of the Gaussian and Eisenstein integers.

Raising these initiators to consecutive rational integer powers results in a power series of families, shown with logarithmic spirals. Representative curves are shown from these higher families. (They are set to the same rotation for illustration purposes).

Only perfect power families allow multiple-length segments in their curves. Higher-power families can inherit features of their root families. As we venture out to explore families with much larger norms, how do things change qualitatively? Curves of large prime families require large generators, since they must contain the same number of segments/integers as the family norm. This means that the *design* of the generator (analogous to "search" in terms of the amount of effort required to obtain it) will become arbitrarily inelegant. In contrast, treasures can be found among perfect power families, where curves can have multiple-length (and thus fewer) segments.

From observations, it appears that non-power composite families (e.g., Gaussian norm 10 and Eisenstein norm 12) cannot have generators with multiple-length segments. The presumed reason is that self-similarity obeys power laws *only*. There may be exceptions to this rule. One curious example is the curve in Figure 16—a member of the Gaussian norm 10 family. Its integers include norms 1, 2, and 5 (all divisors of 10). A close-up of the interior of this curve is shown with splined rendering. It is not known whether any self-crossing or overlapping occurs at higher iterations.

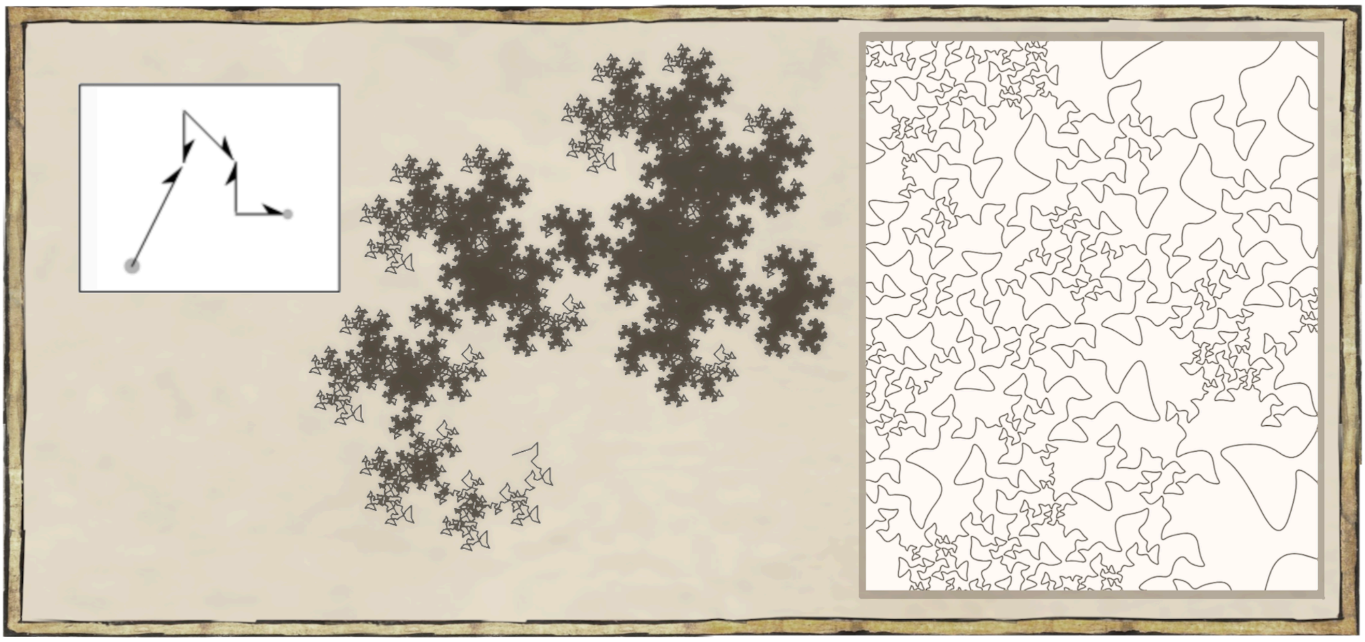

Figure 16. The 10-Dragon of the Gaussian norm 10 family.

Families in Diverse Neighborhoods

Returning to the two slices shown in Figures 8 and 9, let us explore how families of plane-filling curves relate to the structure of the integer rings they occupy. Figure 17 shows several plane-filling curves associated with their norms in their representative slice. Primes are indicated with gray dots. Recall that the norms in this first slice can be represented in all the other slices. The Gaussian integers highlighted with red dots have norms that are powers of 2. These integers have associates that form the power of 2 spiral shown in Figure 15. The Gaussian integer highlighted with a green dot has norm 25. There are 2 norm 25 integers shown in the Gaussian slice (4+3*i*, and 5).

Although their norms are the same, they have different divisibility properties: the one lying on the real line is the product of two integers of norm 5 (2+*i* and its conjugate 2-*i*).

The light blue line connecting the endpoints of each curve is the initiator. It's length is proportional to the distance from 0 to its associated integer. Thus the curves fill different areas in the plane. But the length of the initiator is not the only factor determining the area; the nature of the curve matters, as described in relation to the curves in Figure 10. For instance, the curve labeled G8 spreads out to a large area while G10 is smaller and more compact despite having a larger norm. The reason is that the G8 curve is self-avoiding (lower curvature, less dense, larger area), while the G10 curve is edge-filling (higher curvature, more dense, smaller area).

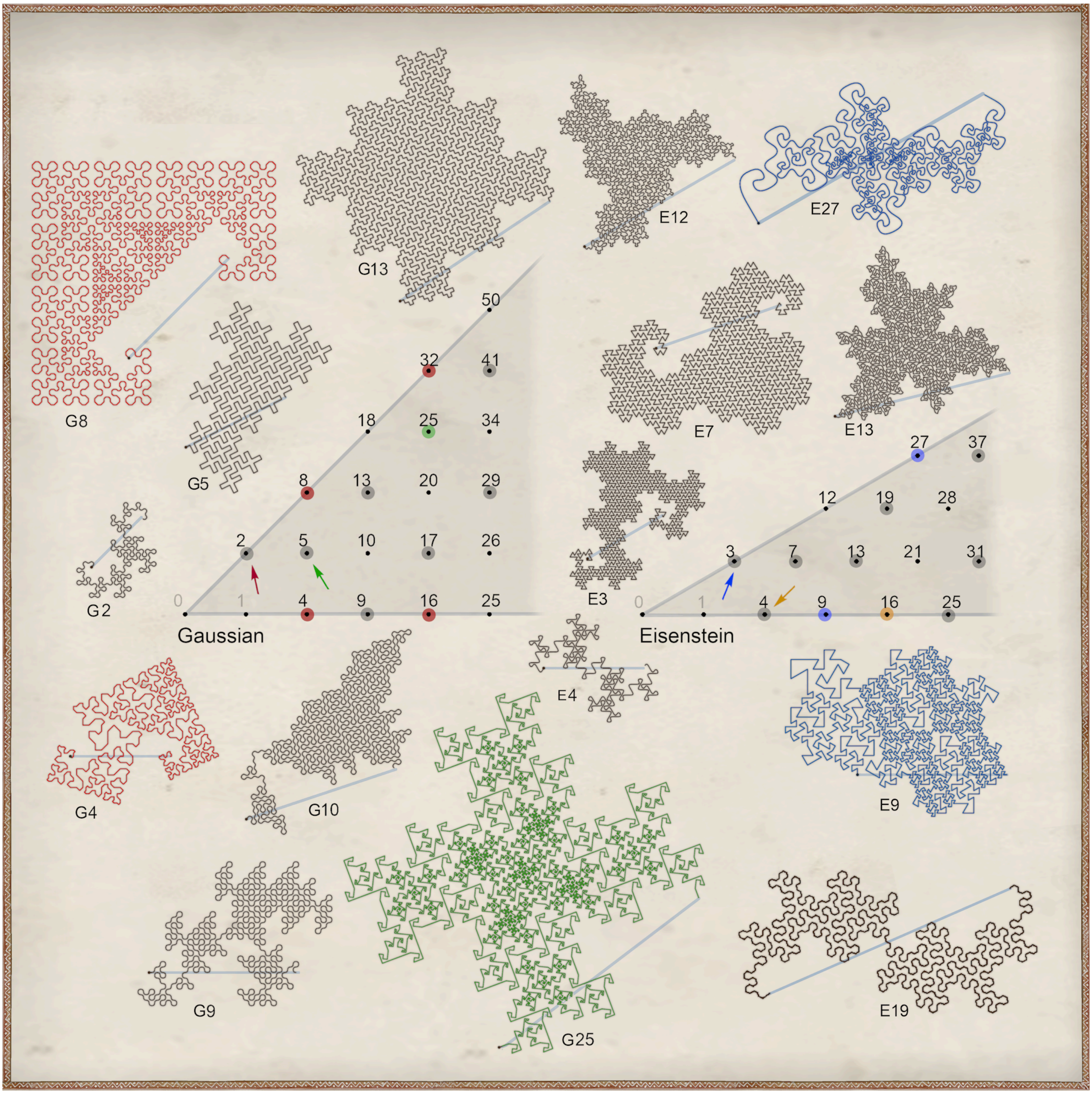

Figure 17. Families of plane-filling curves indicated in the Gaussian and Eisenstein lattices, with example curves.

The real number line (main axis) is where we find families with plane-filling curves having regular polygonal boundaries (squares or regular triangles). McKenna (1994) describes several families on the real line as well as those that "perturb" the grid by popping up into the realm of the imaginary axis. The diagonal (secondary axis, or *off-axis*) at the top of the slice is the home of families with curves having right-triangular (Gaussian) or √3 rectangular (Eisenstein) boundaries. It appears that only families on these two axes have curves with straight boundaries (fractal dimension 1).

Visualizing Number

There are many more visual patterns to be discovered in plane-filling curves, and we can be sure that those patterns are not the artwork of insects, mathematicians, or demons; they are the emergent uncrafted residue of crafted dynamical systems whose complexity grows through recursion. Visual number theorists like Katherine Stange (2018) have found fractal patterns that express the complex fabric of algebraic spaces. What new kinds of mathematical patterns can we extract from the flesh of plane-filling curves? The 4-level plane-filling curve in Figure 18 is from the Eisenstein order-49 perfect power family (the curve is splined to separate touch-points in this rendering). The generator is shown at upper-left. It is a variation of the 7-Dragon (a squared 7-Dragon mother with a baby in place of its last segment).

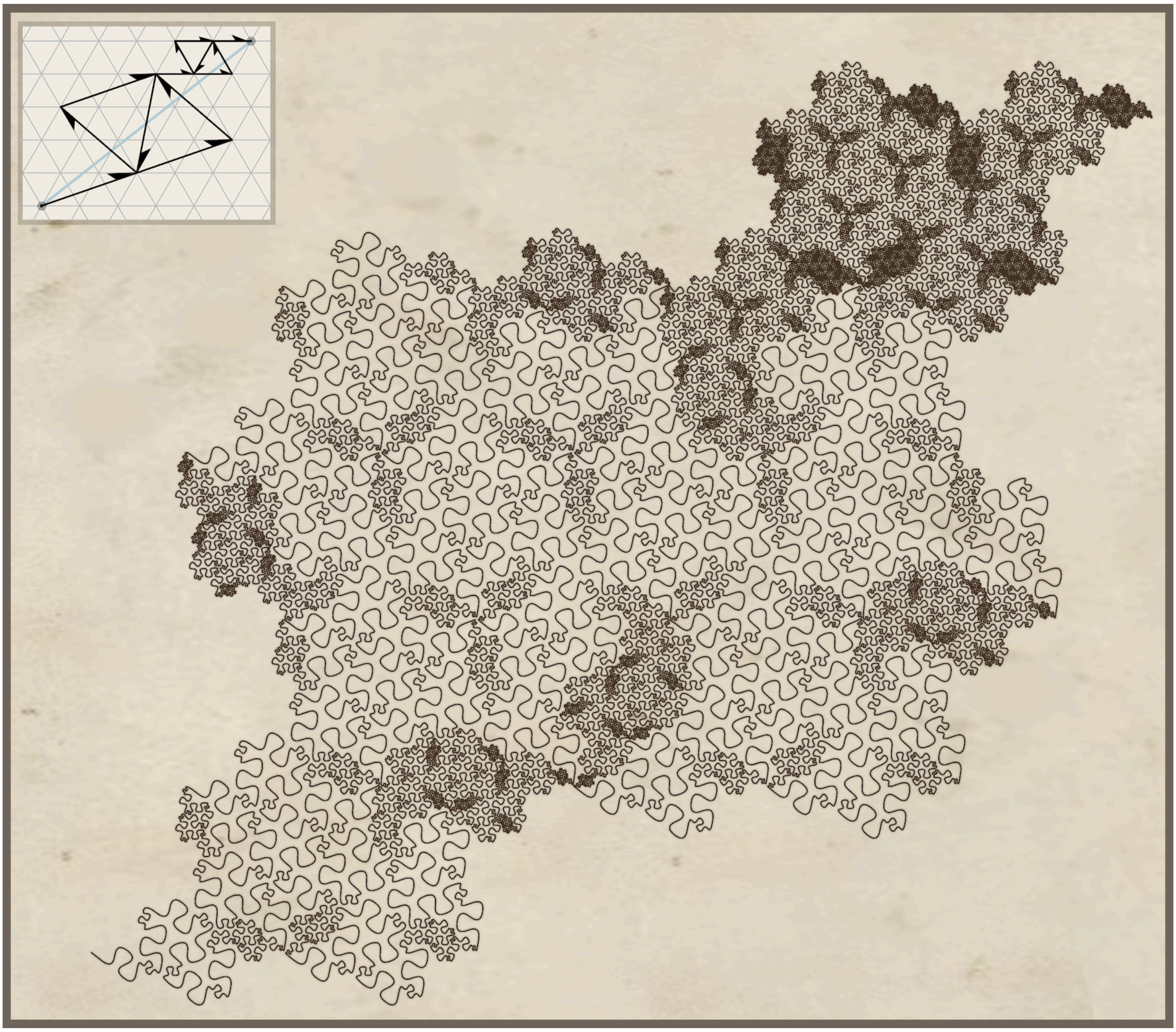

Figure 18: An order-49 plane-filling curve based on the 7-Dragon, with its generator.

The dense clumping from these last 7 segments gets progressively distributed into the flesh with each fractal iteration. The resulting internal patterns visually express powers of 7, as modified by the meandering characteristic shape of the sweep. These are just a few of the many possible observations about the ways that families of plane-filling curves relate to the structure of the complex integers.

Other Lattices

Not all plane-filling curves correspond to the square or triangular lattice. There are some notable exceptions, a few of which are mentiond here. Arndt (2016) developed a technique using an L-system to generate plane-filling curves on non-uniform grids. Figure 19 shows the two required generators for a curve that conforms to an Archimedean tiling (the rhombitrihexagonal 3.4.6.4 tiling), which consists of triangles, squares, and hexagons. This curve can be used to evenly fill the Snowflake Sweep.

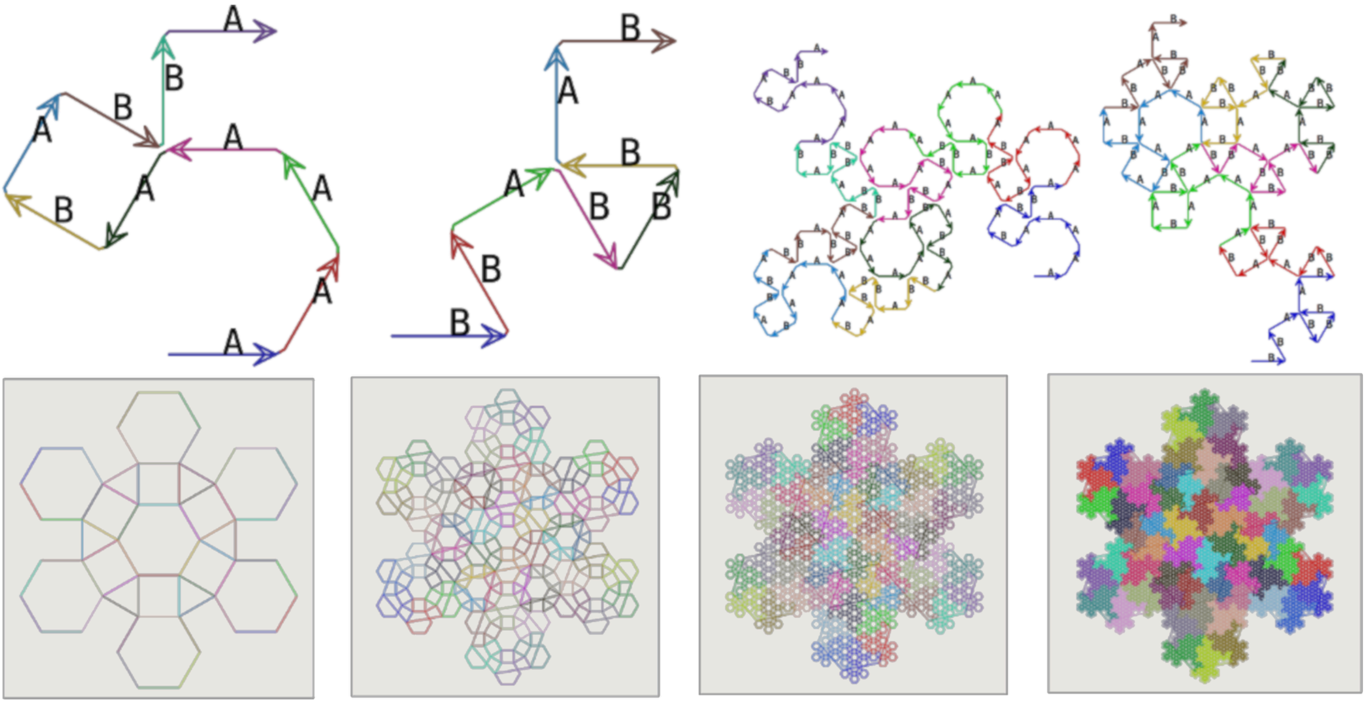

Figure 19.A two-generator curve developed by Arndt that fills the Snowflake Sweep.

Bakker (2025) found a plane-filling curve called the "Duck Curve" (Figure 20). It does not correspond to a square or triangular lattice. It has a scaling factor of √2.

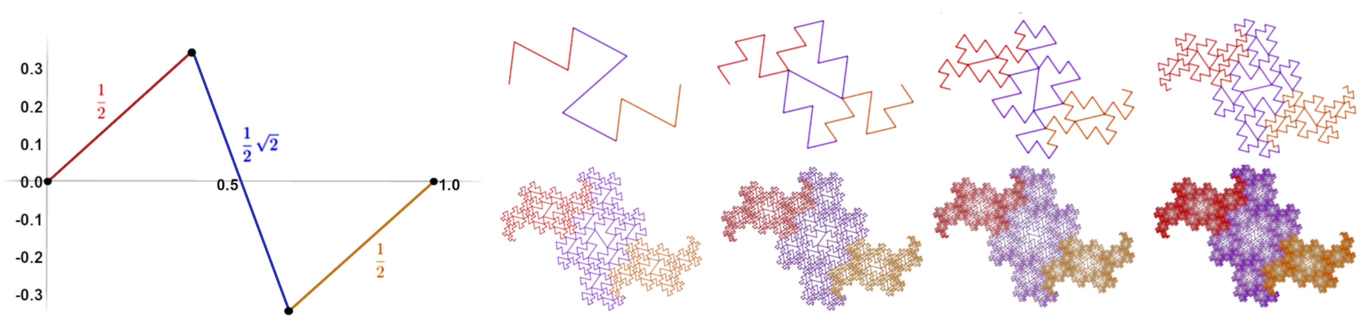

Figure 20. Generation of the Duck Curve.

Haverkort (2022) identified a grid of parallelograms associated with the Duck curve that undergoes translation, rotation, and scaling (shrinking) with each successive iteration. Bill Gosper (2023) independently identified a transformation from the TerDragon to a curve identical to the Duck Curve, as shown in Figure 21. Intermediate stages are self-crossing.

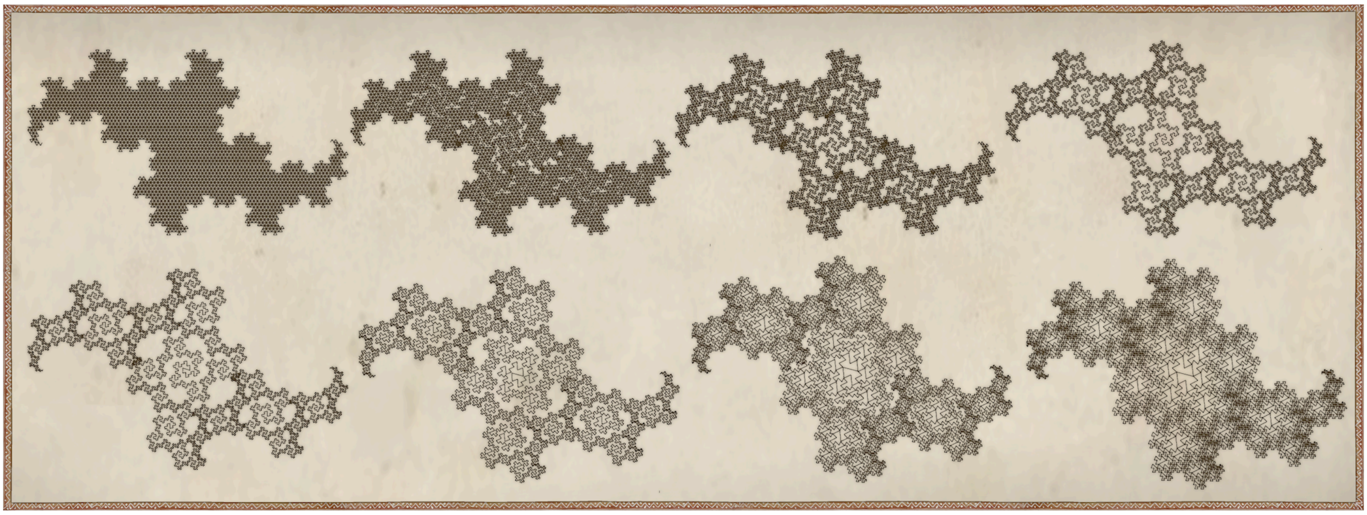

Figure 21. Transformation from TerDragon to Duck Curve.

Aperiodic Plane-filling Curves

Regular triangles, squares and hexagons tile the plane, while pentagons notoriously defy monotiling. "Pieces of pentagons" (several kinds of polygons whose angles are multiples of 36 degrees and whose edge lengths have golden ratios) can fill the plane—and they can create aperiodic patterns, as demonstrated by Penrose tilings (Penrose, 1974). Ramachandrarao, et.al (2000) showed that Penrose tilings can be used to generate aperiodic plane-filling curves. A pentagon-filling curve by Henle (2021) is shown in Figure 22(a).

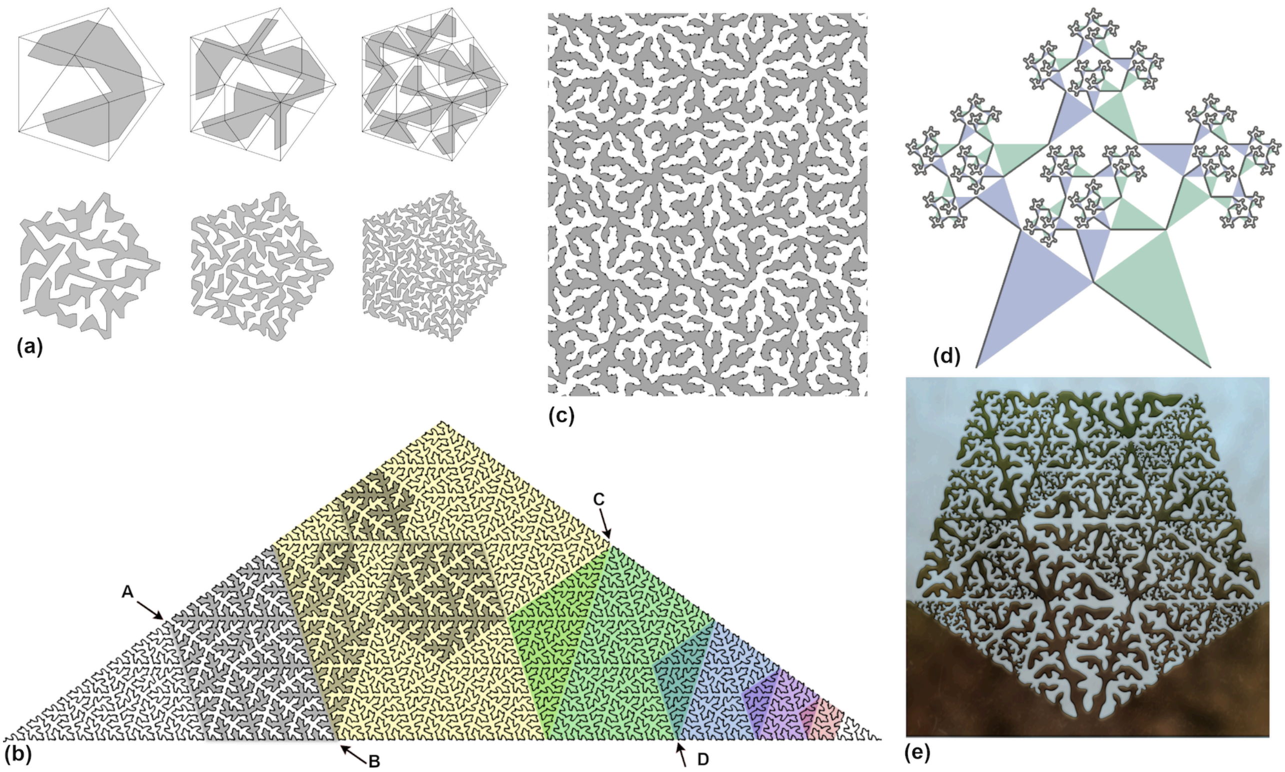

Figure 22. (a) The Pautze Curve; (b) Three curves on a golden triangle initiator

Pautze (2021) and Ventrella (2021) developed techniques for generating plane-filling curves based on the roots of unity in the 10th cyclotomic field. Figure 22(b) shows a colored rendering of the Pautze curve, which requires two generators. It has regular pentagons embedded everywhere within. Each internal pentagon is made of a contiguous portion of the curve (e.g., from A to B, and from C to D). This curve is modified to separate the touch-points so that its path can be appreciated. Hassell (2014) describes an aperiodic curve based on Ammann A5 Tiles (a portion of this curve, filled-in, is shown in Figure 22(c)). A pentagram design by the author based on a curve with fractal dimension < 2 is shown in Figure 22(d), with added golden triangles, as well as the plane-filling curve shown in Figure 22(e) (Ventrella, 2021). These are generated with a variation of the algorithm that was modified to use cyclotomic integers, and it allows for multiple generators.

Reducing the Search Space

Fishing for plane-filling curves requires knowledge of their ecological niche. They occupy a vanishingly small portion of the space of all possible curves, and the subset of FASS curves is even smaller. As a thought experiment, study the wiggly closed curve in Figure 23(a) (shown with a snake-like Medieval beast, for aesthetic effect). Five snapshots from a simulation are shown at progressions over time. The curve is comprised of a closed chain of point masses connected by spring forces, drawn as line segments. Springs have a constant ideal length. At regular time intervals as the simulation runs, random springs are chosen for insertion of new point-masses, splitting the spring into two springs, which push their constituent point masses apart. To ensure that this is a self-avoiding curve, each point mass has a soft-collision radius, causing any pair of points to push away from each other if their collision radii intersect. The accumulation of collision forces causes the overall shape to gradually grow larger.

This simulation is meant to illustrate the idea that plane-filling curves can, in principle, be generated stochastically, having no symmetry or regularity whatsoever, other than artifacts caused by the nature of the algorithm, such as spring constant, timing, inertia, collision forces, spring length, etc. Such a curve cannot be fully described with an elegant equation or compact set of rules. As it grows larger, its description becomes necessarily larger: it has high *Kolmogorov complexity*. Compare this to the self-avoiding curve in Figure 23(b). Its expression can be shown visually as a generator or as an ordered list of integers with transforms Figure 23(c). The number of iterations (fractal levels) used to make this curve is 5. Increasing this value adds complexity to the curve without increasing the size of its expression. Compared to its generator expression, the resulting plane-filling curve has emergent features that can be described on many levels.

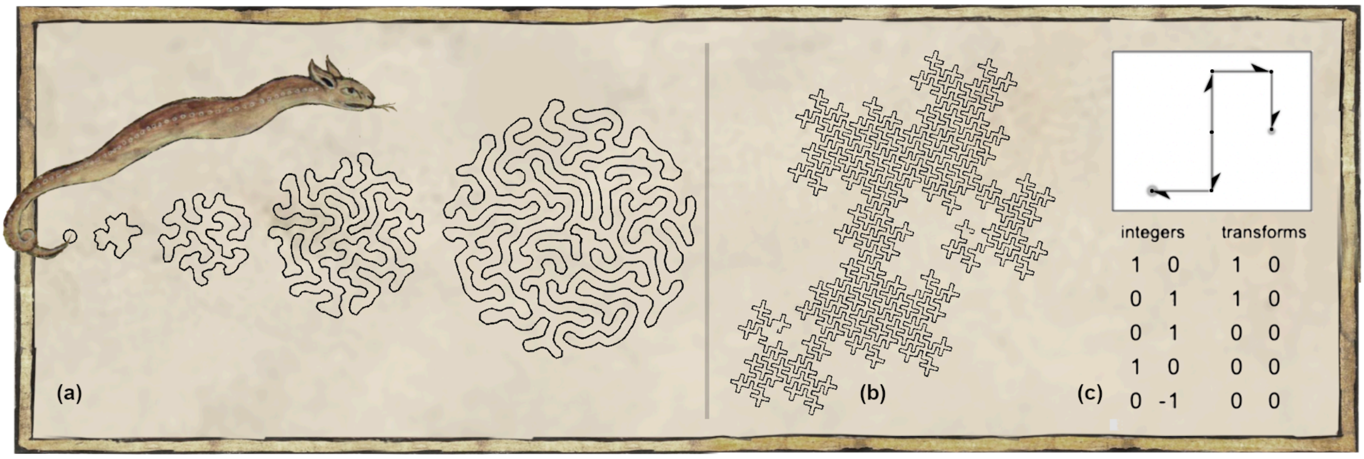

Figure 23 (a) a randomly-generated curve from a physical simulation; (b): a norm-5 curve; (c) its generator expression.

Imagine a search algorithm designed to scan every possible plane-filling curve based on generators with a variable number of segments having any real-number length and connected at any angle. It would have to search a large space indeed. We can see from observation that generators of plane-filling curves typically have angles that are small integer divisions of pi, and most lengths correspond to orthogonals or diagonals of regular polygons (as with cyclotomic integers). This is one rationale for using a lattice frame. Another rationale is that it invites complex integers into the conversation. The proposed expression for a generator that determines a plane-filling curve consists of the following:

1. the ring identifier (Gaussian vs. Eisenstein)
2. an ordered array of complex integers
3. an associated array of transforms.

(The family initiator f could be derived as the sum of the integers). These values determine the shape of the generator and the parameters for iteration. The ring identifier determines how the two terms of the integers map to the complex plane (e.g., $2+i$ for Gaussian vs. $2+\omega$ for Eisenstein). Transforms are represented with 2 bits (rotation and reflection), of which there are 4 combinations, as illustrated in Figure 6(c). The search consists of finding the subset of possible generators on initiator f that admit plane-filling curves. Knowledge about the divisibility characteristics of f helps reduce the size of the search space, resulting in the following rule: f must be the sum of an ordered array of integers (visualized as the segments in the generator), and those integers must all be nth roots of f. The implied sub-rules are:

(1) If f is prime, all integers must be units.
(2) If f is a perfect power, the integers can include a mixture of different nth roots of f.
(3) If f is composite but not a perfect power then the integers must all have the same norm (of one of its divisors).

Here are a few examples: the order-7 family of curves (e.g., Gosper) has generators consisting only of units because its initiator ($3+\omega$) is prime. The order-8 family of curves (e.g., Figure 10 (right) and Figure 17(G8)) has generators consisting of integers whose norms can be combinations of 1, 2, and 4, because its initiator ($2+2i$) is a perfect-power.

Many Questions

In the context of the proposed classification, there are many questions:

- Can any of the examples cited above of plane-filling curves that do not conform to the square or triangular lattice be described with algebraic geometry? Could this classification scheme be revised/expanded to encompass these examples?
- Can any non-perfect-power families that are otherwise composite have any curves with multiple divisors? (The example in Figure 16 begs the question).

- How do the patterns found in these plane-filling curves compare and contrast to those generated using complex bases, as described by Davis and Knuth?
- Can the parallelogram lattice implied in the Duck Curve be described using complex integers? Could the present scheme be expanded to include it?
- How can the proposed system of classification and generation be combined with the cyclotomic-based methods cited? What generalizations can be made?
- Can any of the following attributes be categorized or systematically-described in terms of algebraic geometry?
    1. area (inversely proportional to density and average curvature)
    2. skin (fractal dimension and other properties of the boundary)
    3. topology type (disk, kissing disk-chain, pseudo-gasket, etc.)
    4. symmetry type (bilateral / order of reflection; n-fold rotational symmetry; equilateral polygon: real axis, etc.)

Summary/Conclusion

Self-similar plane-filling fractal curves have recursive, hierarchical structure at all scales. They have some things in common with biological growth and form, such as the generation of emergent complexity from a relatively simple initial starting point: the generator. For math novices with an appetite for geometry, fractal curve generators present themselves as visual / conceptual puzzles. This becomes a project leading to continual discovery. The algebraic properties of complex integers provides rich context for categorizing these properties. This chapter presents a cursory exploration into the subject, with the goal of inspiring students, geometry-lovers, and mathematicians to continue the search.

As a final visual treat, three plane-filling curves are presented. Figure 24(a) is a filled-in curve by McKenna (2007) called "Thirteenski"; Figure 24(b) is the "Gosper Dragon", having a belly in the shape of the Gosper Island (Ventrella, 2019); Figure 24(c): shows the 7-segment generator for the Gosper Dragon; and Figure 24(c)) is a curve by Arndt (2016).

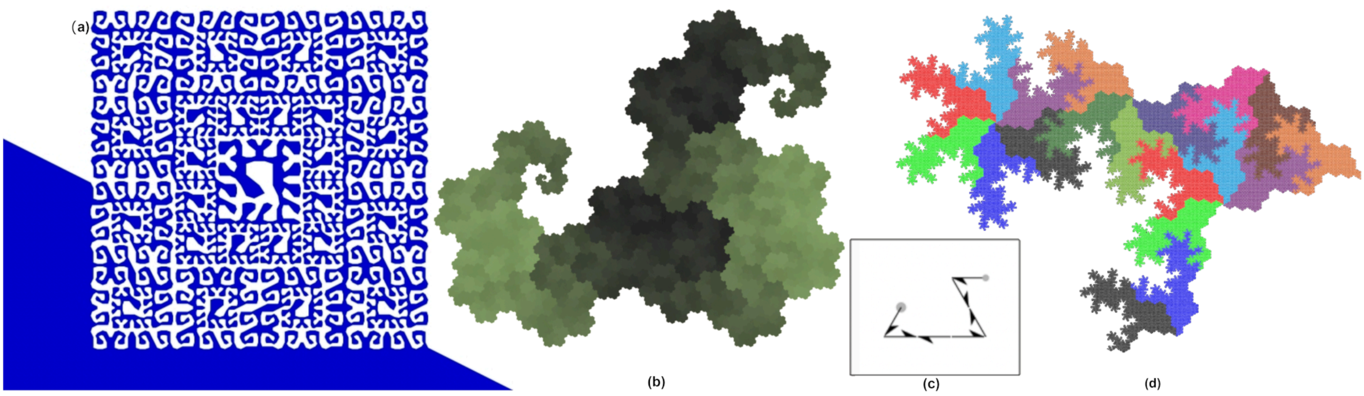

Figure 24: (a) "Thirteenski"; (b) Gosper Dragon; (c) Gosper Dragon generator; (d) an Arndt curve

References

Abelson H, diSessa A (1986) Turtle Geometry. MIT Press

Arndt J, Handl J (2019) Puzzling Plane-Filling Curves. Bridges 2019 Conference Proceedings, Tessellations Publishing

Arndt J (2016) Plane-filling Curves On All Uniform Grids. https://arxiv.org/pdf/1607.02433.pdf. Accessed July 31, 2025. Accessed July 31, 2025

Bakker N (2025) The Duck Curve: https://wisbaksels.nl/publicaties/. Accessed July 31, 2025

Barnsley, M. F. (1988). Fractals everywhere. Academic Press.

Bogomolny A (2018) Plane Filling Curves: All Peano Curves. https://www.cut- the-knot.org/Curriculum/Geometry/PeanoComplete.shtml. Accessed July 31, 2025

Bourke P (2001) Julia Set Fractal. http://www.paulbourke.net/fractals/juliaset/. Accessed July 31, 2025

Carbajo V (2007) Fractals. https://www.carbajo.net/varios/fractales-i.html. Accessed July 31, 2025

Chang A, Zhang T (2000) On the Fractal Structure the Boundary of Dragon Curve. J. Recreational Mathematics 30 (1999–2000), 9–22.

Davis C, Knuth D (1970) Number Representations and Dragon Curves. Journal of Recreational Mathematics. 3, 66-81, 133-149

Dawkins R. (1986). The Blind Watchmaker. New York: W. W. Norton & Company, Inc.

Dekking, M Paperfolding Morphisms, Planefilling Curves, and Fractal Tiles. Theoretical Computer Science, 414, (2012) 20-37

Fathauer, R. (2025) Iterating Polyominoes to Create Fractal Reptiles. Bridges 2025 Conference Proceedings. Tessellations Publishing.

Fathauer R (2013) Twenty Generations of Dragons. https://mathartfun.com/Gallery.html. Accessed July 31, 2025

Fatou, P (1917) "Sur les substitutions rationnelles", Comptes Rendus de l'Académie des Sciences de Paris, vol. 164, pages 806–808 and vol. 165, pages 992–995.

Fukuda M, Shimizu G, Nakamura (2001) New Gosper Space Filling Curves, Proceedings of the International Conference on Computer Graphics and Imaging (CGIM2001) 34–38.

Gardner M (1967) Mathematical Games. Scientific American, April 1967, Vol. 216, No. 4, pp. 118–120.

Gilbert W (1982) Fractal Geometry Derived from Complex Bases. The Mathematical Intelligencer. vol. 4.

Gosper B (2023). (personal communication).

Harriss E, Stange, K E, Trettel S (2022) Algebraic Number Starscapes. Experimental Mathematics, 31(4), 1098–1149. https://doi.org/10.1080/10586458.2022.2102094

Hassell R (2014) A Plane-Filling Curve Using Ammann A5 Tiles. Proceedings of Bridges 2014: Mathematics, Music, Art, Architecture, Culture

Haverkort H (2022) Nico Bakker's duck plane-filling curve. http://herman.haverkort.net/_media/research/e

Henle F (2016) Space-filling curves on non-periodic tilings. G4G12 Exchange Book, volume 1, pages

45–51, Atlanta, Georgia, 2016. Gathering 4 Gardner, Inc., Gathering 4 Gardner, Inc. URL as of April 2021: https://www.gathering4gardner.org/g4g12-exchange-book/.

Hilbert D (1891) "Über die stetige Abbildung einer Linie auf ein Flächenstück", Math. Ann. (38), pp. 459-460.

Julia, G (1918) "Mémoire sur l'iteration des fonctions rationnelles", Journal de Mathématiques Pures et Appliquées, vol. 8, pages 47–245.

Khmelnik, S. I. (1964). Specialized digital computer for operations with complex numbers. Questions of Radio Electronics, Series RVE, Issue 2, pp. 20–26. (In Russian)

Mandelbrot B (1977) The Fractal Geometry of Nature. W. H. Freeman and Company

McKenna D (1994) SquaRecurves, E-Tours, Eddies and Frenzies: Basic Families of Peano Curves on the Square Grid. Guy, Richard K., Woodrow, Robert E.: The Lighter Side of Mathematics: Proceedings of the Eugene Strens Memorial Conference on Recreational Mathematics and its History, pp. 49-73, Mathematical Association of America

McKenna D (2007). The Mathematical Art of Doug McKenna. https://mathemaesthetics.com/MathArtPrints.html. Accessed July 31, 2025

McKenna D (2024) The Art of Space-Filling Domino Curves. Bridges 2024 Conference Proceedings. Tessellations Publishing.

McKenna D (2019) The Art of New Generalized Hilbert Curves: Pinwheel, Mandala, Meta-Hilbert, and Tendril-Based Half-Domino Curves. https://www.jointmathematicsmeetings.org/amsmtgs/2245_abstracts/1154-k1-1300.pdf. Accessed July 31, 2025

Pautze S (2021) Space-Filling, Self-Similar Curves of Regular Pentagons, Heptagons and other n-gons. Bridges 2021 Conference Proceedings. Tessellations Publishing.

Peano G (1890) Sur une courbe, qui remplit toute une aire plane, Mathematische Annalen 36 (1): 157–160

Peitgen, H. O, and Richter, P. H. (1986). The Beauty of Fractals: Images of Complex Dynamical Systems. Heidelberg: Springer-Verlag.

Penney, W. F. (1965). A “binary” system for complex numbers. Journal of the ACM (JACM), 12(2), 247–248.

Pickover, C. A. (1990) Computers, Pattern, Chaos, and Beauty. New York: St. Martin’s Press.

Prusinkiewicz P, Lindenmayer A. (1990) The Algorithmic Beauty of Plants. Springer

Ramachandrarao, Arvind Sinha and D. Sanyal (2000) On the fractal nature of Penrose tiling. Current

Science, vol. 79, no. 3, 10 August 2000.

Stange K (2018) Visualizing the Arithmetic of Imaginary Quadratic Fields. International Mathematics Research Notices, Volume 2018, Issue 12, 13 June 2018, Pages 3908–3938

Ventrella J (2012) Brainfilling Curves - A Fractal Bestiary. Eyebrain Books/Lulu Press

Ventrella J (2019) The Family Tree of Fractal Curves. Eyebrain Books/Lulu Press

Ventrella, J (2019) Portraits from the Family Tree of Plane-filling Curves. Proceedings of Bridges 2019: Mathematics, Art, Music, Architecture, Education, Culture.

Ventrella J (2021) Designing Fractal Curves with Five-Fold Rotational Symmetry Using the Complex Number Golden Ratio. Proceedings of Bridges 2019: Mathematics, Art, Music, Architecture, Education, Culture.

Verrill H (2024) L-systems for the boundaries of fractal dragon space filling folding curves. https://arxiv.org/pdf/2402.16106. Accessed July 31, 2025

Wilson E.O. (1984) Biophilia. Harvard University Press